111. A Calculus of the Tensor Product of Two Holonomic Systems with Support on Non-singular Plane Curves

By Shinichi TAJIMA
Faculty of General Education, Niigata University

(Communicated by Kôsaku YOSIDA, M. J. A., Dec. 14, 1987)

The aim of this paper is to calculate (in the framework of $\mathcal{D}_X$-Modules) the tensor product of two holonomic systems supported on non-singular plane curves.

§0. **Notation.** Let X be a domain in C^2 containing the origin $P=(0,0)$. Let $\mathcal{O}_X$ be the sheaf of germs of holomorphic functions and $\mathcal{D}_X$ the sheaf on X of rings of linear partial differential operators of finite order with holomorphic coefficients. Let F be an analytic plane curve (on X) passing through P with a defining equation $f=0$. Let us denote by $\mathcal{H}^1_{[F]}(\mathcal{O}_X)$ the sheaf of algebraic local cohomology with supports in F:

$$\mathcal{H}^1_{[F]}(\mathcal{O}_X)=\varinjlim_k \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_X/(f)^k,\ \mathcal{O}_X)=\mathcal{O}_X[f^{-1}]/\mathcal{O}_X.$$

Note that the module $\mathcal{H}^1_{[F]}(\mathcal{O}_X)$, which is endowed with a natural structure of left $\mathcal{D}_X$-Module, is a holonomic system.

§1. **Statement of the results.** Let F and G be plane curves meeting properly at a point P. We set:

$$\mathcal{L}=\mathcal{H}^1_{[F]}(\mathcal{O}_X)\hat{\otimes}\mathcal{H}^1_{[G]}(\mathcal{O}_X)$$
$$=\mathcal{D}_{X\times X}\otimes_{p_1^{-1}\mathcal{D}_X\otimes p_2^{-1}\mathcal{D}_X}(p_1^{-1}\mathcal{H}^1_{[F]}(\mathcal{O}_X)\otimes p_2^{-1}\mathcal{H}^1_{[G]}(\mathcal{O}_X)),$$

where p_1 and p_2 are the first and the second projections from $X\times X$ to X. The following quasi-isomorphism is a special case of a result of Kashiwara [2]:

$$\mathcal{H}^1_{[F]}(\mathcal{O}_X)\overset{L}{\otimes}_{\mathcal{O}_X}\mathcal{H}^1_{[G]}(\mathcal{O}_X)=\mathcal{D}_{X\to X\times X}\overset{L}{\otimes}_{\mathcal{D}_{X\times X}}\mathcal{L}.$$

we have the following

Theorem 1 (Intersection formula). *Let F and G be non-singular plane curves (on X) intersecting properly at P. We assume $F\cap G=P$. Then we have the following isomorphisms of $\mathcal{D}_X$-Modules.*

(1) $\mathcal{T}or_k^{\mathcal{D}_{X\times X}}(\mathcal{D}_{X\to X\times X},\mathcal{L})=0$ *for* $k\neq 0$,

(2) $\mathcal{H}^1_{[F]}(\mathcal{O}_X)\otimes_{\mathcal{O}_X}\mathcal{H}^1_{[G]}(\mathcal{O}_X)=\mathcal{D}_{X\to X\times X}\otimes_{\mathcal{D}_{X\times X}}\mathcal{L}=\mathcal{H}^2_{[P]}(\mathcal{O}_X)$,

where $\mathcal{H}^2_{[P]}(\mathcal{O}_X)$ is the $\mathcal{D}_X$-Module of algebraic local cohomology with supports in P.

Remark 2. In the case where F and G being transversal the results above are well known (cf. Sato–Kawai–Kashiwara [3], Schapira [4]).

Example 3. Set $X=\{(x,y)\in C^2\}$, $X_1=\{(x_1,y_1)\in C^2\}$, and $X_2=\{(x_2,y_2)\in C^2\}$. X_1 and X_2 are two copies of X. Put $F=\{(x_1,y_1)\,|\,y_1=0\}$, $G=\{(x_2,y_2)\,|\,y_2-x_2^2=0\}$. We denote by $\delta(y_1)$ (resp. $\delta(y_2-x_2^2)$) the canonical generator of

$\mathcal{H}^1_{[F]}(\mathcal{O}_X)$ (resp. $\mathcal{H}^1_{[G]}(\mathcal{O}_X)$). We set:

$$m = l_{X\to X_1\times X_2}\otimes(\delta(y_1)\hat{\otimes}\delta(y_2-x_2^2))$$

where $l_{X\to X_1\times X_2}$ is a canonical section of $\mathcal{D}_{X\to X_1\times X_2}$ (cf. [3], [4]). We get:

$$\mathcal{D}_X m = \mathcal{D}_{X\to X_1\times X_2}\otimes_{\mathcal{D}_{X_1\times X_2}}(\mathcal{H}^1_{[F]}(\mathcal{O}_{X_1})\hat{\otimes}\mathcal{H}^1_{[G]}(\mathcal{O}_{X_2}))$$

and

$$\mathcal{D}_X m = \mathcal{D}_X\Big/\left(\mathcal{D}_X x^2 + \mathcal{D}_X\left(x\frac{\partial}{\partial x}+2\right)+\mathcal{D}_X y\right).$$

Setting $u=-2xm$, we have

$$\mathcal{D}_X m = \mathcal{D}_X u = \mathcal{D}_X/(\mathcal{D}_X x + \mathcal{D}_X y) = \mathcal{H}^2_{[P]}(\mathcal{O}_X),$$

where $P=(0,0)$.

Theorem 4 (Self intersection formula). *Let F be a non-singular plane curve. We set:*

$$\mathcal{F} = \mathcal{D}_{X\times X}\otimes_{p_1^{-1}\mathcal{D}_X\otimes p_2^{-1}\mathcal{D}_X}(p_1^{-1}\mathcal{H}^1_{[F]}(\mathcal{O}_X)\hat{\otimes}p_2^{-1}\mathcal{H}^1_{[F]}(\mathcal{O}_X)).$$

Then we have

(1) $\mathcal{T}or_k^{\mathcal{D}_{X\times X}}(\mathcal{D}_{X\to X\times X},\mathcal{F})=0 \quad k\neq 1$

(2) $\mathcal{T}or_1^{\mathcal{D}_{X\times X}}(\mathcal{D}_{X\to X\times X},\mathcal{F})=\mathcal{H}^1_{[F]}(\mathcal{O}_X)$.

§2. Sketch of the proofs. Set $X_1=\{(x_1,y_1)\in C^2\}$, $X_2=\{(x_2,y_2)\in C^2\}$, and $X=\{(x,y)\in C^2\}\cong\{(x_1,y_1,x_2,y_2)\in X_1\times X_2 \mid x_1=x_2,\ y_1=y_2\}$. Denoting the canonical section of $\mathcal{D}_{X\to X_1\times X_2}$ by $l_{X\to X_1\times X_2}$ we have:

$$(*)\qquad \begin{cases} x l_{X\to X_1\times X_2} = l_{X\to X_1\times X_2}\otimes x_1 = l_{X\to X_1\times X_2}\otimes x_2 \\ \dfrac{\partial}{\partial x} l_{X\to X_1\times X_2} = l_{X\to X_1\times X_2}\otimes\left(\dfrac{\partial}{\partial x_1}+\dfrac{\partial}{\partial x_2}\right) \\ \text{etc.} \end{cases}$$

Set $\mathcal{L}=\mathcal{H}^1_{[F]}(\mathcal{O}_{X_1})\hat{\otimes}\mathcal{H}^1_{[G]}(\mathcal{O}_{X_2})$. Recall that $\mathcal{D}_{X\to X_1\times X_2}\overset{L}{\otimes}\mathcal{L}$ is quasi-isomorphic to the following complex:

$$(**)\qquad 0\longleftarrow \mathcal{L}\xleftarrow{(x_1-x_2,\ y_1-y_2)} \begin{matrix}\mathcal{L}\\ \otimes \\ \mathcal{L}\end{matrix} \xleftarrow{\begin{pmatrix} y_1-y_2 \\ -x_1+x_2\end{pmatrix}} \mathcal{L}\longleftarrow 0.$$

By using the relations $(*)$ we can calculate the $\mathcal{D}_X$-Module structure of the homology groups of the complex $(**)$. This yields the results.

References

[1] M. Kashiwara: On the maximally overdetermined system of linear differential equations. I. Publ. RIMS, Kyoto Univ., **10**, 563–579 (1975).

[2] ——: On the holonomic systems of linear differential equations. II. Inventiones Math., **49**, 121–135 (1978).

[3] M. Sato, T. Kawai, and M. Kashiwara: Microfunctions and pseudo-differential equations. Lecture Notes in Math., vol. 287, pp. 265–529 (1973).

[4] P. Schapira: Microdifferential Systems in the Complex Domain. Springer-Verlag (1985).