106. *Uniform Distribution of the Zeros of the Riemann Zeta Function and the Mean Value Theorems of Dirichlet L-functions*

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We shall give a brief survey of some applications of our previous works on the uniform distribution of the zeros of the Riemann zeta function $\zeta(s)$ (cf. [1], [2]). The details will appear elsewhere. We assume the Riemann Hypothesis throughout this article.

Let γ run over the positive imaginary parts of the zeros of $\zeta(s)$. We may recall the following two theorems which are special cases of the more general theorem in the author's [2]. The first theorem is a refinement of Landau's theorem (cf. [5]). We put $\Lambda(x)=\log p$ if $x=p^k$ with a prime number p and an integer $k\geqq 1$ and $\Lambda(x)=0$ otherwise.

Theorem 1. *For any positive α,*

$$\sum_{0<\gamma\leqq T} e^{i\alpha\gamma}=-\frac{1}{2\pi}\frac{\Lambda(e^\alpha)}{e^{\alpha/2}}T+\frac{e^{i\alpha T}}{2\pi i\alpha}\log T+O\left(\frac{\log T}{\log\log T}\right).$$

The second theorem gives us a connection of the distribution of γ with a rational number.

Theorem 2. *For any positive α,*

$$\lim_{T\to\infty}\frac{1}{T}\sum_{0<\gamma\leqq T} e^{i\gamma\log(\gamma/2\pi e\alpha)}=\begin{cases}-e^{(1/4)\pi i}\dfrac{C(\alpha)}{2\pi} & \text{if } \alpha \text{ is rational}\\ 0 & \text{if } \alpha \text{ is irrational,}\end{cases}$$

where we put $C(\alpha)=\dfrac{1}{\sqrt{\alpha}}\dfrac{\mu(q)}{\varphi(q)}$ with the Möbius function $\mu(q)$ and the Euler function $\varphi(q)$ if $\alpha=a/q$ with relatively prime integers a and $q\geqq 1$.

We should remark that the remainder terms in Theorems 1 and 2 depend on α heavily. In our applications with which we are concerned here it is necessary and important to clarify the dependences on α. In fact, if we follow the proofs of our theorems above in pp. 103-112 of [2], then we get the following explicit versions of them.

Theorem 1′. *Let $0<Y_0<Y\leqq T$. Then*

$$\sum_{Y_0<\gamma\leqq Y} e^{i\alpha\gamma}=A(\alpha,Y,Y_0)+O((\alpha e^{(1/2)\alpha}+1)\log Y/\log\log Y)$$
$$-\frac{\alpha}{2\pi}\sum_{k=2}^{\infty}\frac{\Lambda(k)}{k^{1+\delta}\log k}e^{(1/2+\delta)\alpha}\int_{Y_0}^{Y} e^{-it\log k+it\alpha}dt$$
$$-\frac{\alpha}{2\pi}\sum_{k=2}^{\infty}\frac{\Lambda(k)}{k^{1+\delta}\log k}e^{-(1/2+\delta)\alpha}\int_{Y_0}^{Y} e^{it\log k+it\alpha}dt$$

uniformly for a positive α, where we put $\delta=1/\log T$ and

$$A(\alpha, Y, Y_0)=\begin{cases}\dfrac{e^{i\alpha Y}}{2\pi i\alpha}\log\dfrac{Y}{2\pi}-\dfrac{e^{i\alpha Y_0}}{2\pi i\alpha}\log\dfrac{Y_0}{2\pi}+0\left(\mathrm{Min}\left(\dfrac{\log Y}{\alpha^2},\ \dfrac{1}{\alpha^3}\right)\right)\\ or\\ 0\left(\dfrac{\log Y}{\alpha}\right).\end{cases}$$

Theorem 2′. *Let m and n be integers satisfying $1\leqq m\leqq n$ and q be an integer $\geqq 1$. Suppose that $2\pi n^2/q\leqq Y\leqq T$. Then*

$$\begin{aligned}\sum_{2\pi n^2/q<\gamma\leqq Y} & e^{i\gamma\log(q\gamma/2\pi emn)}\\ &=-e^{(1/4)\pi i}\sqrt{\frac{mn}{q}}\sum_{n/m<k<Yq/2\pi mn}\Lambda(k)e^{-2\pi imnk/q}\\ &+0\left(\sqrt{\frac{mn}{q}}\sum_{n/m\leqq k\leqq n/m(1-\varepsilon)}\Lambda(k)\right)\\ &+0\left(\sqrt{\frac{mn}{q}}\sum_{Yq/(1+\varepsilon)2\pi mn\leqq k\leqq Yq/2\pi mn}\Lambda(k)\right)\\ &+0\left(\sqrt{\frac{Yq}{mn}}(T^{2/5}+(\log qT)^4)\right)\\ &+\delta_{n,m}0\left(\frac{n}{\sqrt{q}}\log Y\right)+(1-\delta_{n,m})0\left(\frac{\log Y}{\log(n/m)}\right),\end{aligned}$$

where we put $\varepsilon=T^{-2/5}$, $\delta_{n,m}=1$ if $m=n$ and $\delta_{n,m}=0$ otherwise.

We now state what kind of applications we have in mind. Our first application is to refine Gonek's result in [3] and [4] which states that

$$\begin{aligned}\sum_{0<\gamma\leqq T}&\left|\zeta\left(\frac{1}{2}+i\left(\gamma+\frac{2\pi\alpha}{\log(T/2\pi)}\right)\right)\right|^2\\ &=\left(1-\left(\frac{\sin\pi\alpha}{\pi\alpha}\right)^2\right)\frac{T}{2\pi}\log^2 T+0(T\log T),\end{aligned}$$

where $T>T_0$ and α is a real number satisfying $|\alpha|\leqq\dfrac{1}{4\pi}\log\dfrac{T}{2\pi}$. Now using Riemann-Siegel formula for $\zeta(s)$ (cf. 4.17.4 of [5]) and Theorems 1′ and 2′ above with $q=1$, we get the following theorem.

Theorem 3. *Suppose that $T>T_0$ and $\Delta=\dfrac{2\pi\alpha}{\log(T/2\pi)}(\neq 0)$ is bounded. Then*

$$\begin{aligned}\sum_{0<\gamma\leqq T}&\left|\zeta\left(\frac{1}{2}+i\left(\gamma+\frac{2\pi\alpha}{\log(T/2\pi)}\right)\right)\right|^2\\ &=\left(1-\left(\frac{\sin\pi\alpha}{\pi\alpha}\right)^2\right)\frac{T}{2\pi}\log^2\frac{T}{2\pi}\\ &+2\left(-1+C_0+(1-2C_0)\frac{\sin 2\pi\alpha}{2\pi\alpha}+\mathrm{Re}\left(\frac{\zeta'}{\zeta}(1+i\Delta)\right)\right)\frac{T}{2\pi}\\ &\times\log\frac{T}{2\pi}+G(T,\alpha)+0(T^{9/10}\log^2 T),\end{aligned}$$

where C_0 is the Euler constant and $G(T,\alpha)$ will be described below.

$$
\begin{aligned}
G(T,\alpha)=&-\frac{T}{\pi}\operatorname{Re}\Big\{C_0-1+\frac{\zeta'}{\zeta}(1+i\Delta)+(1+i\Delta)\int_1^\infty\frac{R(y)}{y^{2+i\Delta}}dy\\
&+2\int_1^\infty\frac{R_1(y)}{y^2}dy+2^{2+i\Delta}(T/2\pi)^{(1/2)i\Delta}\sum_{l=1}^\infty(\pi l)^{i\Delta}\int_{2\pi l}^\infty\frac{\cos w}{w^{3+i\Delta}}dw\\
&-2\Big(\zeta(1+i\Delta)-\frac{1}{i\Delta}\Big)\frac{(T/2\pi)^{(1/2)i\Delta}}{2+i\Delta}+2\Big(\zeta(1+i\Delta)-\frac{1}{i\Delta}-C_0\Big)\\
&\times\Big(\frac{(T/2\pi)^{(1/2)i\Delta}-1}{i\Delta}-\frac{(T/2\pi)^{(1/2)i\Delta}}{(1+i\Delta)i\Delta}\Big)+\frac{5}{6}\frac{(T/2\pi)^{(1/2)i\Delta}}{2+i\Delta}+\frac{1}{1+\Delta^2}\\
&+(2C_0-1)((T/2\pi)^{(1/2)i\Delta}-(T/2\pi)^{i\Delta})/(1+i\Delta)\\
&+\Big(\zeta^2(1+i\Delta)+\frac{1}{\Delta^2}-\frac{2C_0}{i\Delta}\Big)(T/2\pi)^{i\Delta}/(1+i\Delta)\Big\},
\end{aligned}
$$

where we put

$$R(y)=\sum_{n\le y}\Lambda(n)-y$$

and

$$R_1(y)=\sum_{n\le y}\sum_{k|n}\Lambda(k)k^{i\Delta}+y\frac{\zeta'}{\zeta}(1-i\Delta)-\frac{y^{1+i\Delta}}{1+i\Delta}\zeta(1+i\Delta)$$

and remark that $R_1(y)\ll y^{1/2+\varepsilon}$ for any positive ε and $G(T,\alpha)\ll T$.

Our second application is to show the following theorem.

Theorem 4. *Let $L(s,\chi)$ be a Dirichlet L-function with a primitive character $\chi \bmod q\ge 2$. Suppose that $q\ll(\log T)^A$ with an arbitrarily large constant A. Then we have*

$$
\begin{aligned}
&\sum_{0<\gamma\le T}L\Big(\frac{1}{2}+i\gamma,\chi\Big)\\
&\quad=\frac{T}{2\pi}\Big\{-L(1,\bar\chi)\chi(-1)\tau(\chi)\frac{\mu(q)}{\varphi(q)}+\frac{L'}{L}(1,\chi)\Big\}\\
&\qquad+O(T\exp(-C_1\sqrt{\log qT})),
\end{aligned}
$$

where we put

$$\tau(\chi)=\sum_{n=1}^{q}\chi(n)e^{2\pi in/q}$$

and C_1 is some positive absolute constant.

In particular, we obtain the following corollary which expresses a connection of the distribution of γ with the values of $L(s,\chi)$ at $s=1$.

Corollary. *Let $L(s,\chi)$ and q be given as above. Then we have*

$$
\begin{aligned}
&\lim_{T\to\infty}\frac{2\pi}{T}\sum_{0<\gamma\le T}L\Big(\frac{1}{2}+i\gamma,\chi\Big)\\
&\quad=-L(1,\bar\chi)\chi(-1)\tau(\chi)\frac{\mu(q)}{\varphi(q)}+\frac{L'}{L}(1,\chi).
\end{aligned}
$$

In a similar manner we can obtain various mean value theorems like

$$\sum_{0<\gamma\le T}\zeta'\Big(\frac{1}{2}+i\gamma\Big)$$

and

$$\sum_{0<\gamma\le T}\Big|L\Big(\frac{1}{2}+i\gamma,\chi\Big)\Big|^2.$$

Here we mention only the the following theorem.

Theorem 5.

$$\sum_{0<\gamma\leqq T}\zeta'\left(\frac{1}{2}+i\gamma\right)=\frac{1}{4\pi}T\log^2\frac{T}{2\pi}+(C_0-1)\frac{T}{2\pi}\log\frac{T}{2\pi}+\left(C_2-C_3+\frac{1}{2}\right)\frac{T}{2\pi}+0(T^{9/10}\log^2 T),$$

where we put

$$C_2=\int_1^\infty\frac{\{y\}-\frac{1}{2}}{y^2}\log y\,dy,\qquad C_3=\int_1^\infty\frac{1-\log y}{y^2}R(y)\,dy,$$

and $\{y\}$ is the fractional part of y.

This should be compared with Gonek's result in [3] which states that

$$\sum_{0<\gamma\leqq T}\left|\zeta'\left(\frac{1}{2}+i\gamma\right)\right|^2=\frac{1}{24\pi}T\log^4 T+0(T\log^3 T).$$

References

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