98. The Existence of Solvable Operators in the Ideal

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§1. **Introduction and preparation.** Let R be the set of all real numbers, and let f be a function on the plane $R\times R$ taking values in R. Let (x, y) be coordinates in $R\times R$. In this note we announce a property and applications of the following polygonal functional equation for all $(x, y)\in R\times R$, $t\in R$:

$$(P)\quad \begin{aligned} &f(x+t, y)+f(x-t, y)+f(x, y+t)+f(x, y-t)\\ &=f(x+t, y+t)+f(x-t, y+t)+f(x+t, y-t)+f(x-t, y-t). \end{aligned}$$

Let F denote the set of all functions $f: R\times R\to R$. Denote by A the algebra of all linear operators on F. As usual, the multiplication in F is composition of operators. For $t\in R$, $f\in F$, define the shift operators $X^t, Y^t: F\to F$ by $(X^t f)(x, y)=f(x+t, y)$ and $(Y^t f)(x, y)=f(x, y+t)$ for all $x, y\in R$. $S\subset A$ denotes the commutative sub-algebra with unit $(1=X^0=Y^0)$ generated by finite linear combinations of shift operators X^t and Y^t on F. Define the operators $\sigma(t)$ and $\theta(t)$ by $\sigma(t)=X^t+X^{-t}+Y^t+Y^{-t}$ and $\theta(t)=(X^t+X^{-t})(Y^t+Y^{-t})=X^tY^t+X^{-t}Y^t+X^tY^{-t}+X^{-t}Y^{-t}$ for $t\in R$. Then the operator $\mu(t)=\sigma(t)-\theta(t)$ for $t\in R$ is in S. For $f\in F$ the operator equation $(\mu(t)f)(x, y)=0$ reduces to polygonal functional equation (P). It is clear that if f satisfies the equation $(\mu(t)f)(x, y)=0$ for $t\in R$, then f also satisfies the equation $(\nu(t)\mu(t)f)(x, y)=0$ for any $\nu(t)\in S$. Hence we consider the ideal generated in S by the family of operators

$$(Q)\qquad \{\mu(t)\mid t\in R\}.$$

In [1] J. Aczél, H. Haruki, M. A. McKiernan and G. N. Sakovič considered the equivalence of polygonal functional equation (P), or briefly, $\sigma(t)-\theta(t)=0$, and the Haruki square mean value equation ([10], [13])

$$(H)\quad f(x+t, y+t)+f(x-t, y+t)+f(x+t, y-t)+f(x-t, y-t)=4f(x, y)$$

or, simply, $\theta(t)-4=0$ under the assumption that $f: R\times R\to R$ is continuous everywhere; it is not necessarily true without continuity assumption. Further, they proved in [1] that if f is continuous and satisfies (H) for all $(x, y)\in R\times R$, $t\in R$, then f is C^∞ on the plane. Hence the only continuous solution of (H) is a certain harmonic polynomial of bounded degree. So, by the equivalency of (P) and (H), the only continuous solution of (P) is also given by the same harmonic polynomial. Moreover, they also obtained [1] the general solution of (H) when no regularity assumptions are imposed on f. On the other hand, if the general theorem of McKiernan [11, Theorem 2, p. 32] is directly applied to equation (H), then one can readily obtain operators corresponding to an equation which we can solve so that the general solution of (H) is also obtained from those solutions of difference functional equations. However, the general solution of (P) remains open under no regularity assumptions. Equation (P) is not a mean value equation of a type considered by McKiernan in [11]. Hence, it may not be possible to apply the result of McKiernan (ibid., p. 32) to equation (P) in order to determine the general solution of (P). From this point of view, McKiernan has raised the following problem in [12]. Does the ideal generated by (Q) contain an operator corresponding to an equation which we can solve? The aim of this note is to inform an answer to this problem.

§ 2. The existence of solvable operators. Our main result is as follows.

Theorem. *The ideal generated by* (Q) *contains the operators*

$$(Q^*) \qquad \{(X^t-1)^7 \mid t \in R\} \quad \textit{and} \quad \{(Y^t-1)^7 \mid t \in R\}.$$

The proof is based on a reduction from the equation $(\mu(t)f)(x, y)=0$ to difference functional equations which we can solve. By some long algebraic manipulations we can obtain the operator

$$\{X^{8t}+X^{-8t}-4X^{4t}-4X^{-4t}-X^{2t}-X^{-2t}+4X^{t}+4X^{-t} \mid t \in R\}$$

in the ideal from (Q). If the above operator acts on $f(x, y)$ then, by writing $\phi_y(x) := f(x, y)$ for fixed y, we have the equation

$$\phi_y(x+8t)+\phi_y(x-8t)-4\phi_y(x+4t)-4\phi_y(x-4t)-\phi_y(x+2t)$$
$$-\phi_y(x-2t)+4\phi_y(x+t)+4\phi_y(x-t)=0,$$

which is an equation of the form

$$\sum_{i=0}^{n} f_i(x+\alpha_i y)=0,$$

where $f_i: R \to R$ and $\alpha_i \neq 0$ for $i=0, 1, \cdots, n$ and $\alpha_j \neq \alpha_k$ for $j \neq k$, considered in [5, p. 116]. Hence, by Lemma 3.1 of [5] we obtain the following two difference functional equations

$$(D) \qquad [(X^t-1)^7 f](x, y)=0 \quad \text{and} \quad [(Y^t-1)^7 f](x, y)=0$$

for all $(x, y) \in R \times R$, $t \in R$.

§ 3. General and regular solutions of (P). The partial difference operator $\underset{x,t}{\triangle}$ in the x direction with increment t is defined by $\underset{x,t}{\triangle}=(X^t-1)$. Similarly, define $\underset{y,t}{\triangle}$ by $\underset{y,t}{\triangle}=(Y^t-1)$. Then it follows from the above theorem that if $f: R \times R \to R$ satisfies equation (P) for all $(x, y) \in R \times R$, $t \in R$, then by (D) f also satisfies both

$$(3.1) \qquad (\underset{x,t}{\triangle}^7 f)(x, y)=0 \quad \text{and} \quad (\underset{y,t}{\triangle}^7 f)(x, y)=0$$

for all $(x, y) \in R \times R$, $t \in R$. Hence, the results of Z. Ciesielski [3] or J. H. B. Kemperman [7] (among others) imply from (3, 1) that

Corollary 3.1. *Let f of (P) on an open interval J be continuous at one point of J, or measurable on J, or, bounded on some set of positive measure in J. Then f is C^∞ on J, and hence the only solution of (P) is a harmonic polynomial of degree at most* 4.

We note that our theorem also yields the structure of the most general solution of equation (P) (hence, if not C^∞, then unbounded on any set of positive measure). Hence, it readily follows from (3.1) and the general results of S. Mazur and W. Orlicz [8] or M. A. McKiernan [9] (for more details of the type $(\triangle_t^n f)(x)=0$, where $\triangle_t := E^t - 1$, $(E^t f) = f(x+t)$, and generalized polynomials, see H. Cartan [2], E. Hille and R. S. Philips [6]) that

Corollary 3.2. *Without any regularity assumption, if f satisfies equation (P) for all $(x, y) \in R \times R$, $t \in R$, then f is represented by the sum of diagonalized multiadditive symmetric function in x, for each y, and similarly in y, for each x.*

§ 4. Applications of (P). One of them is an application of (P) to a geometric characterization of quadratic functions from the standpoint of conformal-mapping properties. From this point of view H. Haruki [4] gave an application of (H) under two invariant geometric properties on f. Here, if two slightly different invariant geometric properties are imposed on f, then by conformal-mapping properties one can obtain the functional equation

(4.1) $$f(z+te^{\omega i})-f(z-te^{\omega i})=e^{2\omega i}[f(z+te^{-\omega i})-f(z-te^{-\omega i})],$$

for all $z \in C$ (the set of all complex numbers), $t \in R$, and $\omega = \sin^{-1}(1/\sqrt{5})$, where $f: C \to C$ is a continuous function on the complex plane. Then one can show that, by some transformations, both the real and the imaginary parts u, v of f satisfy equation (P). So the above Corollary 3.1 implies continuous functions $u, v \in C^1$. Further, one obtains the Cauchy-Riemann equations so that f can be extended as an entire function of z. (4.1) implies $f'''(z)=0$ in the plane.

The other is the following: if a continuous function $f: R \times R \to R$ satisfies polygonal functional equation (P) for all $(x, y) \in R \times R$, $t \in R$, then also each one of

(H) $$[(\theta(t)-4)f](x, y)=0,$$

(1) $$[(2\theta(t)-4-\sigma(t))f](x, y)=0,$$

(2) $$[(\sigma(t)-4)f](x, y)=0,$$

(3) $$[(\sigma(2t)-\sigma(t))f](x, y)=0,$$

(4) $$[(\theta(t)+\sigma(2t)-8)f](x, y)=0,$$

(5) $$[(\theta(t)+\sigma(t)-8)f](x, y)=0$$

for all $(x, y) \in R \times R$, $t \in R$ and conversely so that they are equivalent to each other. Hence, $f \in C^\infty$ and is a harmonic polynomial of degree at most 4.

In conclusion we note that the above algebraic interpretation and all functional equations considered in this note admit a geometric interpretation.

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