72. Systems of Microdifferential Equations with Involutory Double Characteristics

Propagation Theorem for Sheaves in the Framework of Microlocal Study of Sheaves

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§1. **Introduction.** We study a class of microdifferential equations with double involutory characteristics. Explicitly, let M be a real analytic manifold with a complex neighborhood X and let $\mathfrak{M}$ be a coherent $\mathcal{E}_X$ module defined in a neighborhood of $\rho_0 \in T^*_M X \backslash M$. (See M. Sato *et al.* [4] and P. Schapira [5] for $\mathcal{E}_X$.) We assume that the characteristic variety of $\mathfrak{M}$ is written in a neighborhood of ρ_0 as

(1) $$ch(\mathfrak{M})=\{\rho \in T^*X\,;\, p(\rho)=0\}$$

by a homogeneous holomorphic function p defined in a neighborhood of ρ_0. Here p satisfies the following conditions (2), (3) and (4).

(2) $$p \text{ is real valued on } T^*_M X.$$

(3) $\Sigma=\{\rho \in T^*_M X\backslash M\,;\, p(\rho)=0,\ dp(\rho)=0\}$ is a regular involutory submanifold of $T^*_M X$ of codimension 2 through ρ_0.

(4) $$\mathrm{Hess}\,(p)(\rho) \text{ has rank 1 if } \rho \in \Sigma.$$

In §5, we give a propagation theorem of sheaves in the framework of Microlocal Study of Sheaves due to M. Kashiwara and P. Schapira [2], which will play a powerful role in studying the propagation of singularities for microdifferential systems.

§2. **Notation.** To state the results, we give some prerequisites about 2-microfunctions.

Let Λ be a complexification of Σ in T^*X. Then $\tilde{\Sigma}$ denotes the union of all bicharacteristic leaves of Λ issued from Σ. M. Kashiwara introduced the sheaf $\mathcal{C}^2_{\tilde{\Sigma}}$ of 2-microfunctions along Σ on $T^*_{\Sigma}\tilde{\Sigma}$. By $\mathcal{C}^2_{\tilde{\Sigma}}$, we can study the properties of microfunctions on Σ precisely. Actually, we have exact sequences

(5) $$0\longrightarrow \mathcal{C}_{\tilde{\Sigma}}|_{\Sigma}\longrightarrow \mathcal{B}^2_{\tilde{\Sigma}}\longrightarrow \pi_{\Sigma*}(\mathcal{C}^2_{\tilde{\Sigma}}|_{T^*_{\Sigma}\tilde{\Sigma}\backslash\Sigma})\longrightarrow 0 \qquad (\pi_{\Sigma}: T^*_{\Sigma}\tilde{\Sigma}\backslash\Sigma\longrightarrow\Sigma)$$

and

(6) $$0\longrightarrow \mathcal{C}_M|_{\Sigma}\longrightarrow \mathcal{B}^2_{\tilde{\Sigma}}.$$

Here $\mathcal{B}^2_{\tilde{\Sigma}}=\mathcal{C}^2_{\tilde{\Sigma}}|_{\Sigma}$ and $\mathcal{C}_{\tilde{\Sigma}}$ is the sheaf of microfunctions along $\tilde{\Sigma}$. Moreover, we have a canonical spectral map

(7) $$Sp^2_{\Sigma}: \pi_{\Sigma}^{-1}(\mathcal{C}_M|_{\Sigma})\longrightarrow \mathcal{C}^2_{\tilde{\Sigma}},$$

by which we define the 2-singular spectrum for $u \in \mathcal{C}_M|_{\Sigma}$ as

(8) $$SS^2_{\Sigma}(u)=\mathrm{supp}\,(Sp^2_{\Sigma}(u)).$$

We can identify

$$(9) \qquad T_{\Sigma}^{*}\tilde{\Sigma} \xrightarrow{\sim} \bigcup_{\Gamma} T^{*}\Gamma$$

where the union in the right side is taken for all bicharacteristic leaves of Σ. For any C^1 function g defined in an open subset Ω of $T_{\Sigma}^{*}\tilde{\Sigma}$, we define a vector field on Ω by

$$(10) \qquad H_g^{rel} = H_{\Gamma}(dg|_{T^*\Gamma})$$

where $H_{\Gamma}: T^*T^*\Gamma \to TT^*\Gamma$ is Hamiltonian isomorphism. We remark here that H_g^{rel} is tangent to $T^*\Gamma$ for any leaf Γ. See also N. Tose [8] for another description of H_g^{rel} and refer to M. Kashiwara and Y. Laurent [1] and Y. Laurent [3] for more details about 2-microfunctions.

§ 3. **Statement of the result.** We set for $\rho \in \Sigma$ and $\tau \in T_{\Sigma}^{*}\tilde{\Sigma}|_{\rho}$

$$(11) \qquad p_{\Sigma}(\rho, \tau) = \langle \text{Hess}\,(p)(\rho) \cdot H_{\Sigma}(\tau), H_{\Sigma}(\tau) \rangle.$$

Here H_{Σ} is Hamiltonian isomorphism

$$(12) \qquad H_{\Sigma}: T_{\Sigma}^{*}\tilde{\Sigma} \longrightarrow T_{\Sigma}T_{M}^{*}X.$$

To state the main theorems, we give

Lemma 1. *We decompose p_{Σ} as*

$$(13) \qquad p_{\Sigma} = p_0 \cdot p_1^2$$

by real analytic functions p_0 and p_1 satisfying

$$(14) \qquad p_0 \neq 0 \quad \text{on } T_{\Sigma}^{*}\tilde{\Sigma} \backslash \Sigma.$$

By the function p_1 above, we can state

Theorem 2. *Let u be a section of $\mathcal{H}om_{\mathcal{E}_X}(\mathfrak{M}, \mathcal{C}_M)$ defined in a neighborhood of ρ_0. Then $SS_{\Sigma}^2(u)\backslash\Sigma$ is contained in $\{p_1 = 0\}$. Moreover $SS_{\Sigma}^2(u)\backslash\Sigma$ is invariant under $H_{p_1}^{rel}$.*

By Theorem 2 above, we have

Theorem 3. *Let u be a section of $\mathcal{H}om_{\mathcal{E}_X}(\mathfrak{M}, \mathcal{C}_M)$ defined in a neighborhood of ρ_0. Then* supp$(u) \cap \Sigma$ *is a union of projection by π_{Σ} of integral curves of $H_{p_1}^{rel}$ in $\{p_1 = 0\}\backslash\Sigma$.*

Remark 4. In case Hess (p) (ρ) has rank 2, see N. Tose [6], [7], [8], [9] and [10]. Moreover, the codimension of Σ may be taken greater than 2 in [8], [9] and [10].

§ 4. **Proof of Theorem 2.** By finding a suitable real quantized contact transformation, we may assume from the beginning that

$$(15) \qquad ch(\mathfrak{M}) = \{(z, \zeta)\,;\, \zeta_1^2 + A(z, \zeta')\zeta_2^2 = 0\}$$

where $A(z, \zeta)$ is a homogeneous holomorphic function of degree 0 defined in a neighborhood of $\rho_0 = (0, \sqrt{-1}dx_n) \in \sqrt{-1}T^*\boldsymbol{R}^n$. Here we take a coordinate of $\sqrt{-1}T^*\boldsymbol{R}^n$ [resp. $T^*\boldsymbol{C}^n$] as $(x, \sqrt{-1}\zeta \cdot dx)$ [resp. $(z, \zeta \cdot dz)$] with $x, \xi \in \boldsymbol{R}^n$ [resp. $z, \zeta \in \boldsymbol{C}^n$] and set $\zeta' = (\zeta_2, \cdots, \zeta_n)$. Moreover we may assume that

$$(16) \qquad A(z, \zeta')|_{\zeta_2=0} = 0.$$

Here we have in this case

$$(17) \qquad \Sigma = \{(x, \xi)\,;\, \xi_1 = \xi_2 = 0\} \quad \text{and} \quad \Lambda = \{(z, \zeta)\,;\, \zeta_1 = \zeta_2 = 0\}.$$

Then when we put $N = \boldsymbol{C}^2_{(z_1, z_2)} \times \boldsymbol{R}^{n-2}_{(x_3, \cdots, x_n)}$ in $\boldsymbol{C}^n$, we have

$$(18) \qquad \tilde{\Sigma} \xrightarrow{\sim} T_N^* X$$

and $\mathcal{C}_{\tilde{\Sigma}}$ is nothing but the sheaf of microfunctions with holomorphic parameters (z_1, z_2) defined by

$$(19) \qquad \mathcal{C}_{\tilde{\Sigma}} = \mu_N(\mathcal{O}_{\boldsymbol{C}^n})[n-2].$$

Here $\mu_N(\cdot)$ is the functor of Sato's microlocalization along N. (See Kashiwara-Schapira [2] for its definition.) We take a coordinate of $\tilde{\Sigma}$ as $(z', x''; \sqrt{-1}\xi''\cdot dx)$ with x'', $\xi'' \in \boldsymbol{R}^{n-2}$ and $z' \in \boldsymbol{C}^2$, and set that of $T^*\tilde{\Sigma}$ as $(z', x''; \sqrt{-1}\xi''; z'^*dz' + x''^*dx'' + \sqrt{-1}\xi''^*d\xi'')$ with $z'^* = (z_1^*, z_2^*) \in \boldsymbol{C}^2$ and $x''^* = (x_3^*, \cdots, x_n^*)$ and $\xi''^* = (\xi_3^*, \cdots, \xi_n^*) \in \boldsymbol{R}^{n-2}$.

Moreover, in the case above, we have

(20) $$C_{\Sigma}^2 = \mu_{\Sigma}(C_{\tilde{\Sigma}})[2] \xrightarrow{\sim} \mu\,\mathrm{Hom}\,(\boldsymbol{Z}_{\Sigma}, C_{\tilde{\Sigma}})[2]$$

and

(21) $$\boldsymbol{R}\,\mathcal{H}om_{\pi^{-1}(\mathcal{E}_X|_{\Sigma})}(\pi^{-1}(\mathfrak{M}|_{\Sigma}), C_{\Sigma}^2) = \mu\,\mathcal{H}om\,(\boldsymbol{Z}_{\Sigma}, \mathcal{F})[2] \qquad (\pi : T_{\Sigma}^*\tilde{\Sigma} \longrightarrow \Sigma)$$

where $\mathcal{F} = \boldsymbol{R}\,\mathcal{H}om_{\mathcal{E}_X}(\mathfrak{M}, C_{\tilde{\Sigma}})$. (See [2] for the definition of bifunctor $\mu\,\mathcal{H}om\,(\cdot, \cdot)$.) The microsupport of $\mathcal{F}$ can be calculated by Theorem 10.5. 1. of [2] as

(22) $$SS(\mathcal{F}) \subset C_{T_{\Sigma}^*\tilde{\Sigma}}(ch(\mathfrak{M})) = \{\rho \in T^*\tilde{\Sigma} ; z_1^*(\rho) = 0\}.$$

(See Chapter 1 of [2] for the definition of $C_{T_{\Sigma}^*\tilde{\Sigma}}(\cdot)$.) Remarking

(23) $$SS(\boldsymbol{Z}_{\Sigma}) = T_{\Sigma}^*\tilde{\Sigma} \subset \{\mathrm{Re}\, z_1^* = 0\} \quad \text{and} \quad SS(\mathcal{F}) \subset \{\mathrm{Re}\, z_1^* = 0\},$$

we can conclude by the following Theorem 5 that for any

$$u \in H^j(\boldsymbol{R}\,\mathcal{H}om_{\pi^{-1}(\mathcal{E}_X|_{\Sigma})}(\pi^{-1}(\mathfrak{M}|_{\Sigma}), C_{\Sigma}^2)),$$

supp (u) is invariant under ∂/∂_{x_1}. *(q.e.d. for Theorem 2)*

§5. Sheaf theoretical propagation of singularities. The following theorem is essentially due to M. Kashiwara and P. Schapira, which plays a powerful role to study propagation of singularities for microdifferential systems. Here the author would like to gratify to Prof. M. Kashiwara and Prof. P. Schapira for letting me announce the theorem here.

Let X be a C^2 manifold and $D(X)$ denotes the derived category of complexes of $\boldsymbol{Z}$ modules on X and $D^+(X)$ [resp. $D^b(X)$] denotes the full subcategory of $D(X)$ consisting of complexes with cohomologies bounded from below [resp. bounded].

Then we have

Theorem 5. *Let W be an involutory submanifold of T^*X and let $F \in D^+(X)$ and $G \in D^b(X)$. If we assume*

(24) $$SS(F) \subset W \quad \text{and} \quad SS(G) \subset W,$$

then for any $u \in H^j(\mu\,\mathcal{H}om\,(G, F))$, supp (u) is a union of bicharacteristic leaves of W.

Sketch of proof. By the technique of adding one variable due to M. Kashiwara, we may assume from the beginning that W is regular. Moreover, if we find a suitable quantized contact transformation, we may suppose

(25) $$W = \{(x, \xi dx) \in T^*X ; \xi_1 = \cdots = \xi_d = 0\} \qquad (d < n)$$

where we take a coordinate of T^*X as $(x, \xi dx)$ with x and $\xi \in \boldsymbol{R}^n$. Then we have by [2]

(26) $$SS(\mu\,\mathcal{H}om\,(F, G)) \subset C(SS(F), SS(G)).$$

(See Chapter 1 of [2] for the definition of *normal cone* $C(\cdot, \cdot)$.) Then, the right side of (26) is included in

(27) $$\{(x, \xi ; x^*, \xi^*) \in T^*T^*X ; x_1^* = \cdots = x_d^* = 0\}$$

where we take a coordinate of T^*T^*X as $(x, \xi; x^*, \xi^*)$ with x^* and $\xi^* \in \boldsymbol{R}^n$ and identify TT^*X with T^*T^*X by Hamiltonian isomorphism. By (27), we can apply Proposition 4.1.2 of [2] and can verify the assertion of the theorem. (*q.e.d.*)

References

[1] M. Kashiwara and Y. Laurent: Théorèmes d'annulation et deuxième microlocalisation. Prépublication d'Orsay, Université de Paris-Sud (1983).

[2] M. Kashiwara and P. Schapira: Microlocal study of sheaves. Astérisque, **128** (1985).

[3] Y. Laurent: Théorie de la deuxième microlocalisation dans le domaine complexe. Progress in Math., Birkhäuser, no. 53 (1985).

[4] M. Sato, T. Kawai, and M. Kashiwara: Hyperfunctions and pseudo-differential equations. Lect. Notes in Math., Springer, no. 287, pp. 265–429 (1973).

[5] P. Schapira: Microdifferential systems in the complex domain. Grundlehren der Math., Springer, vol. 269 (1985).

[6] N. Tose: On a class of microdifferential equations with involutory double characteristics. J. Fac. Sci., Univ. of Tokyo, **33**, 619–634 (1986).

[7] ——: The 2-microlocal canonical form for a class of microdifferential equations and propagation of singularities. Publ. of RIMS, Kyoto Univ., **23**, 101–116 (1987).

[8] ——: 2nd Microlocalisation and conical refraction. Ann. Inst. Fourier, **37-2** (1987) (in press).

[9] ——: On a class of 2-microhyperbolic systems (to appear in J. Math. pures et appl.).

[10] ——: 2nd microlocalisation and conical refraction (II) (preprint).