60. *On the Value of the Dedekind Sum*

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Let p and q be relatively prime positive integers. The n^{th} *Dedekind sum for* p, q will be defined by

$$S_n(p, q)=\sum_{k=1}^{q-1}\left[\frac{kp}{q}\right]^n \qquad (n=1, 2, \cdots),$$

where $[x]$ denotes, as usual, the greatest integer not exceeding x. It is easy to see that $S_1(p, q)=S_1(q, p)=\frac{1}{2}(p-1)(q-1)$ and the following *reciprocity formulas* are known:

$$(1)\qquad \frac{1}{p}S_2(p, q)+\frac{1}{q}S_2(q, p)=\frac{1}{6pq}(p-1)(2p-1)(q-1)(2q-1),$$

$$(2)\qquad \frac{1}{p(p-1)}S_3(p, q)+\frac{1}{q(q-1)}S_3(q, p)=\frac{1}{4pq}(p-1)(q-1)(2pq-p-q+1)$$

(see, for example, Carlitz [3]).

Assume now $p>q$ throughout this paper. One of the methods to prove these reciprocity formulas is to put $[hq/p]=i-1$ $(i=1, 2, \cdots, q)$ and change $S_n(q, p)$ to the sum with respect to i taking the multiplicities of i's into account. Here the multiplicity of i means the number of h which yields the same value of i and is determined as follows: If h ranges from $[(i-1)p/q]+1$ to $[ip/q]$ for $i<q$, then the value of $[hq/p]$ is $i-1$; for $i=q$, however, h ranges only from $[(q-1)p/q]+1$ to $p-1$. (See, for example, Rademacher and Whiteman [6], (3.5).) Therefore, to obtain the reciprocity relation, we have only to apply the equation

$$(3)\qquad \left[\frac{(h+1)q}{p}\right]-\left[\frac{hq}{p}\right]=\begin{cases}1 & \text{if } h=[ip/q]\ (i=1, \cdots, q-1) \text{ or } p-1,\\ 0 & \text{otherwise.}\end{cases}$$

We have now the following lemma.

Lemma. *Put* $r_1=p-[p/q]q$, *then we get the equation*

$$(4)\qquad \left[\frac{(k+1)p}{q}\right]-\left[\frac{kp}{q}\right]=\begin{cases}[p/q]+1 & \textit{if } k=[jq/r_1]\ (j=1, \cdots, r_1-1) \textit{ or } q-1,\\ [p/q] & \textit{otherwise.}\end{cases}$$

Proof. Substituting $p=[p/q]q+r_1$, we get

$$\left[\frac{(k+1)p}{q}\right]-\left[\frac{kp}{q}\right]=\left[\frac{(k+1)r_1}{q}\right]-\left[\frac{kr_1}{q}\right]+\left[\frac{p}{q}\right].$$

Since q and r_1 are relatively prime and $r_1<q$, the equation (4) follows from the equation (3). □

The equation (4) can be used for reducing the Dedekind sum to a sum of fewer terms and thus for giving an algorithm to evaluate the Dedekind sum in some cases.

Put $r_{-1}=p$, $r_0=q$, and

$$r_i=r_{i-2}-\left[\frac{r_{i-2}}{r_{i-1}}\right]r_{i-1} \qquad \text{for } i=1, 2, \cdots.$$

Then r_i and r_{i+1} are relatively prime, $r_i>r_{i+1}$ for all i, and $r_n=1$ for some $n(\geqq 1)$. Let a_j $(j=-1, 0, \cdots, n-1)$ be inductively defined by the equations

$$a_{-1}=1, \quad a_0=[p/q], \quad \text{and} \quad p=a_jr_j+a_{j-1}r_{j+1}$$

(or equivalently, $a_{j+1}=a_{j-1}+a_j[r_j/r_{j+1}]$), and put $\bar{q}=a_{n-1}$ (so that $1\leqq\bar{q}<p/2$).

Theorem. *Using the above sequence of remainders r_i, $i=1, 2, \cdots, n$, the Dedekind sums for $n=2, 3$ are evaluated as polynomials on p, q, $\bar{q}$, and $[r_{i-1}/r_i]$ $(i=0, 1, \cdots, n)$ as follows:*

(a) $$S_2(p, q)=\frac{1}{3}(p-1)^2(q-1)-\frac{p}{6}(p-q)+\frac{1}{6}\cdot\frac{1-3\cdot(-1)^n}{2}p-\frac{1}{6}$$
$$+\frac{p}{6}\sum_{i=0}^{n}(-1)^i\left[\frac{r_{i-1}}{r_i}\right]+\frac{1}{6}(-1)^n\bar{q},$$

(b) $$S_3(p, q)=\frac{1}{4}(p-1)^3(q-1)-\frac{1}{4}p(p-1)(p-q)+\frac{1}{4}\cdot\frac{1-3\cdot(-1)^n}{2}p(p-1)$$
$$-\frac{p-1}{4}+\frac{1}{4}p(p-1)\sum_{i=0}^{n}(-1)^i\left[\frac{r_{i-1}}{r_i}\right]+\frac{1}{4}(-1)^n(p-1)\bar{q}.$$

Proof. (a) Apply the equation (4) to the sum

$$\sum_{k=1}^{q-1}k^2\left(\left[\frac{(k+1)p}{q}\right]-\left[\frac{kp}{q}\right]\right),$$

then we get

(5) $$\frac{1}{p}S_2(p, q)+\frac{1}{q}S_2(q, r_1)=\frac{1}{6pq}(p-1)(2p-1)(q-1)(2q-1)$$
$$-\frac{1}{6}(q-1)(2q-1)\left[\frac{p}{q}\right].$$

(This equation coincides with the reciprocity formula (1) when $p<q$.)

Since the second term on the left hand side has r_1-1 terms and r_1 is smaller than q, we may regard this as a recursive equation. It follows from the recursive equations

$$\frac{1}{r_i}S_2(r_i, r_{i+1})+\frac{1}{r_{i+1}}S_2(r_{i+1}, r_{i+2})$$
$$=\frac{1}{6r_ir_{i+1}}(r_i-1)(2r_i-1)(r_{i+1}-1)(2r_{i+1}-1)-\frac{1}{6}(r_{i+1}-1)(2r_{i+1}-1)\left[\frac{r_i}{r_{i+1}}\right]$$
$$(i=-1, 0, \cdots, n-2)$$

that

$$\frac{1}{p}S_2(p, q)=\frac{1}{6p}(p-1)^2(q-1)+\sum_{i=-1}^{n-1}\frac{(-1)^{i+1}}{6r_ir_{i+1}}(r_i-1)^2(r_{i+1}-1)^2$$
$$+\sum_{i=0}^{n-1}\frac{(-1)^i}{6}(r_i-1)(r_{i+1}-1).$$

Since we get

$$p\sum_{i=-1}^{n-1}\frac{(-1)^{i+1}}{r_ir_{i+1}}=(-1)^na_{n-1},$$

the equation (a) follows.

Note that

$$(-1)^n q a_{n-1} = 1 + pq \sum_{i=0}^{n-1} \frac{(-1)^{i+1}}{r_i r_{i+1}} \equiv 1 \qquad \text{modulo } p,$$

since $q\sum_{i=0}^{n-1}(-1)^{i+1}/(r_i r_{i+1}) = (-1)^n b_{n-1}$ is an integer where b_j $(j=-1, 0, \cdots, n-1)$ is inductively defined by $b_{-1}=1$, $b_0=[q/r_1]$, and the equation

$$q = b_j r_j + b_{j-1} r_{j+1}.$$

Also note that the integer $\bar{q}=a_{n-1}$ is uniquely determined by the equation $q\bar{q} \equiv (-1)^n$ modulo p and the condition $1 \leqq \bar{q} < p/2$.

(b) Applying the equation (4) to the sum

$$\sum_{k=1}^{q-1} k^3\left(\left[\frac{(k+1)p}{q}\right] - \left[\frac{kp}{q}\right]\right)$$

and using the reciprocity formula

$$\sum_{h=1}^{p-1}\{h^i-(h-1)^i\}\left[\frac{hq}{p}\right]^j + \sum_{k=1}^{q-1}\{k^j-(k-1)^j\}\left[\frac{kp}{q}\right]^i = (p-1)^i(q-1)^j$$

(see Carlitz [3], § 3 and Katase [4], (16)), we obtain the equation

$$S_3(q, p) - S_3(q, r_1) = \frac{1}{4}\left[\frac{p}{q}\right] q^2(q-1)^2.$$

Substituting $p-h$ for h in $S_3(q, p) = \sum_{h=1}^{p-1}[hq/p]^3$ and r_1-j for j in $S_3(q, r_1) = \sum_{j=1}^{r_1-1}[jq/r_1]^3$ and using the recursive equation (5), we obtain the recursive relation

$$\begin{aligned}&\frac{1}{p(p-1)} S_3(p, q) + \frac{1}{q(q-1)} S_3(q, r_1)\\ &\quad = \frac{1}{4pq}(p-1)(q-1)(2pq-p-q+1) - \frac{1}{4} q(q-1)\left[\frac{p}{q}\right].\end{aligned}$$

(This equation coincides with the reciprocity formula (2) when $p<q$.) Following entirely similar method to (a), we get (b). □

Remark 1. The equation (a) plays an important role in classifying 3-dimensional lens spaces by η-invariants (see Katase [5]). On the other hand, it follows from the equation (b) that $4S_3(p, q)$ is divisible by $p-1$. Moreover, analyzing reciprocity formulas, we have the following

Proposition. *$p-1$ divides $S_3(p, q)$ if $p \not\equiv 3$ modulo 4 and never divides $S_3(p, q)$ if $p \equiv 3$ modulo 4; however, $2S_3(p, q)$ is divisible by $p-1$ in this case.*

The equations (a) and (b) are interesting not only with these applications but also as the algorithm for computing these Dedekind sums.

Remark 2. As another application of the equations (3) and (4), we obtain the reciprocity formula

$$\text{(c)} \qquad \sum_{h=1}^{[p/2]}\left[\frac{hq}{p}\right] + \sum_{k=1}^{[q/2]}\left[\frac{kp}{q}\right] = \left[\frac{p}{2}\right]\left[\frac{q}{2}\right]$$

and the value of the *half sum*

$$\text{(d)} \quad \sum_{k=1}^{[q/2]}\left[\frac{kp}{q}\right] = \sum_{i=0}^{n-1}(-1)^i\left[\frac{r_i}{2}\right]\left[\frac{r_{i+1}}{2}\right] + \frac{1}{2}\sum_{i=0}^{n-1}(-1)^i\left[\frac{r_i}{2}\right]\left(\left[\frac{r_i}{2}\right]+1\right)\left[\frac{r_{i-1}}{r_i}\right].$$

In fact, since

$$\sum_{h=1}^{[p/2]}\left(\left[\frac{(h+1)q}{p}\right]-\left[\frac{hq}{p}\right]\right)=\left[\left(\left[\frac{p}{2}\right]+1\right)\frac{q}{p}\right]=\begin{cases}[q/2]+1 & \text{if } p \text{ is even and } p\leqq 2q,\\ [q/2] & \text{otherwise,}\end{cases}$$

we get

$$\left[\frac{(h+1)q}{p}\right]-\left[\frac{hq}{p}\right]=\begin{cases}1 & \text{if } h=[ip/q] \text{ for } i=1,\cdots,\begin{cases}[q/2]+1 & \text{when } p \text{ is even and } p\leqq 2q\\ [q/2] & \text{otherwise}\end{cases}\\ 0 & \text{if } h\neq [ip/q].\end{cases}$$

Hence we get

$$\sum_{h=1}^{[p/2]} h\left(\left[\frac{(h+1)q}{p}\right]-\left[\frac{hq}{p}\right]\right)=\begin{cases}\sum_{i=1}^{[q/2]+1}[ip/q] & \text{if } p \text{ is even and } p\leqq 2q,\\ \sum_{i=1}^{[q/2]}[ip/q] & \text{otherwise.}\end{cases}$$

On the other hand, the left hand side is equal to

$$\sum_{h=1}^{[p/2]}\left((h+1)\left[\frac{(h+1)q}{p}\right]-h\left[\frac{hq}{p}\right]-\left[\frac{(h+1)q}{p}\right]\right)$$
$$=-\sum_{h=1}^{[p/2]}\left[\frac{hq}{p}\right]+\left[\frac{p}{2}\right]\left[\left(\left[\frac{p}{2}\right]+1\right)\frac{q}{p}\right].$$

Hence the extra terms arise on both sides when p is even and $p\leqq 2q$ but they cancel each other and we get the formula (c).

Also applying the equation (4) to the sum $\sum_{k=1}^{[q/2]} k([(k+1)p/q]-[kp/q])$, we obtain the recursive equation

$$\sum_{k=1}^{[q/2]}\left[\frac{kp}{q}\right]+\sum_{j=1}^{[r_1/2]}\left[\frac{jq}{r_1}\right]=\left[\frac{q}{2}\right]\left[\frac{r_1}{2}\right]+\frac{1}{2}\left[\frac{q}{2}\right]\left(\left[\frac{q}{2}\right]+1\right)\left[\frac{p}{q}\right]$$

and hence the equation (d) follows.

Remark 3. As for the Dedekind sums such as $S_n(p, q)$ $(n\geqq 4)$, $\sum_{h=1}^{p-1}(-1)^{h+1+[hq/p]}$ and other five sums of Berndt [2], and $\sum_{h=1}^{p-1}[hq/p]^n\xi^{hq}$ and $\sum_{k=1}^{q-1}[kp/q]^{n-1}\xi^{[kp/q]q}$ (ξ is a p^{th} root of unity) of Katase [4], we have not yet obtained even reciprocity formulas. The foregoing method does not seem to work in these cases.

References

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