66. Canonical Bundles of Compact Complex Surfaces containing Global Spherical Shells

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The purpose of this note is to determine the numerical class of the canonical bundle of a compact complex surface S containing a global spherical shell (GSS for short). Kato [3] first introduced the notion of GSS, and Nakamura [4] classified all surfaces containing GSS's. Nakamura also computed the intersection matrix of curves on such a surface S. In this note, we shall write down the numerical class of the canonical bundle K_S in terms of the intersection matrix. This is one of the problems raised by Dloussky [1].

Details of this note will be published elsewhere.

Notation. Let $A=A_1+\cdots+A_n$ be a linear chain of curves on a surface. Then Zykel(A) denotes $(a_1,\cdots,a_n)$, where the self-intersection number of A_i is $-a_i$.

§1. Let S be a compact complex surface containing a GSS. For the definition of GSS, we refer to Kato [3]. We assume that S has no exceptional curve of the first kind and that the second Betti number $b_2(S)$ is positive. Then, using results of Enoki [2] and Nakamura [4], [5], the surface S is classified as follows.

Theorem 1.1. *S is one of the following surfaces:* (i) *a hyperbolic Inoue surface,* (ii) *a half Inoue surface,* (iii) *a parabolic Inoue surface,* (iv) *an exceptional compactification $S_{n,\beta,t}$ of an affine line bundle on an elliptic curve,* (v) *a* (CB)*-surface.*

The surfaces in the classes (i), (ii) and (iii) are defined in [5], one in the class (iv) is defined in [2], and a (CB)-surface S is defined as follows: S has only finite number of curves, which are rational curves and constitute a single cycle with linear branches sprouting from the cycle.

The surfaces in the classes (i), (ii), (iii) and (iv) have been well studied, and the canonical bundles of them are easily obtained. So, from now on, we let S be a (CB)-surface. The intersection matrix of curves on S was calculated by Nakamura [4] as follows.

Theorem 1.2. *Let C be the set of all curves on S. Then C is decomposed as $C=\sum_{k=1}^{m}(C_k+D_k)$, where*

(i) *(C_k+D_k) has the type $(p_1, q_1, p_2, q_2, \cdots, q_{n-1}, p_n)$,* i.e., *the self-intersection number of components of C_k and D_k are one of the following:*

(1) *if $p_1\geqq 3$, then*

$$\mathrm{Zykel}\,(C_k)=(p_1, \underbrace{2, \cdots, 2}_{q_1-3}, p_2, \underbrace{2, \cdots, 2}_{q_2-3}, \cdots, p_n),$$

$$\mathrm{Zykel}\,(D_k)=(\underbrace{2, \cdots, 2}_{p_n-2}, q_{n-1}, \underbrace{2, \cdots, 2}_{p_{n-1}-3}, q_{n-1}, \cdots, \underbrace{2, \cdots, 2}_{p_1-3})$$

for certain positive integers n ($\geqq 1$), p_n ($\geqq 2$), p_j, q_j ($\geqq 3$, $1 \leqq j \leqq n-1$), *depending on* k,

(3) *if* $p_1=2$, *then*

$$\mathrm{Zykel}\,(C_k)=(\underbrace{2, \cdots, 2}_{q_1-2}, p_2, \underbrace{2, \cdots, 2}_{q_2-3}, p_3, \cdots, p_n)$$

$$\mathrm{Zykel}\,(D_k)=(\underbrace{2, \cdots, 2}_{p_n-2}, q_{n-1}, \underbrace{2, \cdots, 2}_{p_{n-1}-3}, q_{n-2}, \cdots, \underbrace{2, \cdots, 2}_{p_2-3})$$

for certain positive integers n ($\geqq 2$), p_n ($\geqq 2$), p_j, q_j, q_1 ($\geqq 3$, $2 \leqq j \leqq n-1$), *depending on* k ($p_n \geqq 3$ *if* $n=2$).

(ii) $C_k \cdot D_{j-1}=\delta_{jk}$, $C_k \cdot C_{k+1}=1$ ($j, k \in \boldsymbol{Z}/m\boldsymbol{Z}$), *where the last irreducible component of* C_k *and the first of* D_k *meet* C_{k+1} *at distinct points of the first irreducible component of* C_{k+1} *transversally if* $m \geqq 2$. *If* $m=1$, *then* $C_1 \cdot D_1 = 1$, *and the first irreducible component of* C_1 *meets the first of* D_1.

§2. We shall compute the numerical class of the canonical bundle on a (CB)-surface S as above. Here the "numerical class" means the following. The canonical bundle $K_S \in \mathrm{Pic}\, S$ determines an element of $H^2(S, \boldsymbol{Q})$ via

$$\mathrm{Pic}\, S \cong H^1(S, \mathcal{O}_S^*) \longrightarrow H^2(S, Z) \longrightarrow H^2(S, \boldsymbol{Q})$$

and we call this element the numerical class of K_S. As shown by Nakamura [4], a compact complex surface containing a GSS has only finitely many rational curves whose classes then generate $H^2(S, \boldsymbol{Q})$. So the numerical class of K_S can be written as a linear combination of the classes of those curves. From now on, we assume for simplicity that $m=1$ in Theorem 1.2 and we write $C=C_1+D_1$, $C_1=\sum A_i$, and $D_1=\sum B_i$. Let $(p_1, q_1, p_2, \cdots, p_n)$ be the type of $C=C_1+D_1$. The class K_S is written as

$$K_S \equiv s_i A_i + t_i B_i \qquad \text{with } s_i,\ t_i \in \boldsymbol{Q},$$

where $\equiv$ signifies the numerical equivalence, i.e., an equality in $H^2(S, \boldsymbol{Q})$. We shall give an algorithm to compute the numbers s_i, t_i in terms of the integers appearing in the type. One can detect a kind of "strange duality" in the expression of the intersection matrix, and this duality plays an essential role in the computation of the numerical class of K_S. From the viewpoint of the degenerations of $K3$ surfaces (cf. Nishiguchi [6], [7]), we are only interested in case where all s_i and t_i are integers. We assume that K_S is numerically a divisor, which means that all s_i and all t_i are integers. Under these hypotheses, we obtain the following

Theorem 2.1. *The type satisfies one of the following conditions*:

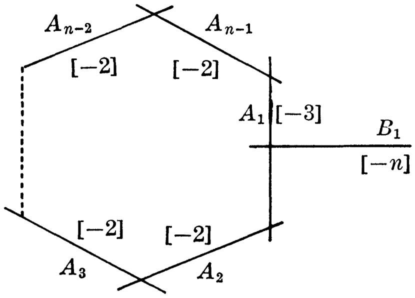
(1) $(3, n, 2)$,

(2) $p_1=2$.

In the case (1), the canonical bundle can be written easily. Namely, we have

Proposition 2.2. *Let* S *be of the type* $(3, n, 2)$. *Then*

$$K_S \equiv -B_1 - 2A_1 - 2A_2 - \cdots - 2A_{n-1};$$



([ ] *indicates the self-intersection number.*)

From now on, we shall restrict ourselves to the case where $p_1=2$. Let S be of the type $(2, q_1, p_2, \cdots, p_n)$. Then K_S can be written as follows in $H^2(S, \boldsymbol{Q})$.

Theorem 2.3. *The self-intersection numbers of the curves A_i and B_i and their coefficients (denoted by* Co( )*) in K_S are given as follows: For C_1 we have*

$$\mathrm{Zykel}(C_1)=(2, \cdots, 2,\ p_2,\ 2, \cdots\cdots\cdots\ p_{n-1},\ 2, \cdots, 2,\ p_n)$$

$$\mathrm{Co}(C_1)=(\cdots, -a_2-d_1, -a_2, -a_2+d_2, \cdots, -a_{n-1}, \cdots, -a_n-d_{n-1}, -a_n)$$

and for D_1 we have

$$\mathrm{Zykel}(D_1)=(2, \cdots, 2,\ q_{n-1},\ 2, \cdots\cdots\cdots\ 2,\ q_2,\ 2, \cdots, 2)$$

$$\mathrm{Co}(D_1)=(\cdots, -b_{n-1}-c_n, -b_{n-1}, -b_{n-1}+c_{n-1}, \cdots, -b_2, -b_2+c_2, \cdots),$$

where the vertical dotted lines signify that the numbers connected by them belong to the same components among A_i's and B_i's, and where $a_2, \cdots, a_n$ and $b_2, \cdots, b_{n-1}$ are positive integers and satisfy the following conditions:

$$(q_i-2)(b_i-1)=a_{i+1}-a_i \qquad (i=2, \cdots, n-1)$$
$$(p_j-2)(a_j-1)=b_j-b_{j-1} \qquad (j=2, \cdots, n)$$
$$b_1=0$$
$$b_n=(q_1-1)-a_n+a_2,$$

and

$$c_i=a_i-1 \qquad (i=2, \cdots, n)$$
$$d_j=1-b_j \qquad (j=2, \cdots, n-1)$$
$$d_1=1$$
$$d_n=a_n-a_2-(q_1-2).$$

Corollary 2.4. *Given $p_2, \cdots, p_n, q_2, \cdots, q_{n-2}$ and a_2, one can determine $a_3, \cdots, a_n, b_3, \cdots, b_n$. Moreover one can write q_1 in terms of $p_2, \cdots, p_n, q_2, \cdots, q_{n-1}$ and a_2.*

Remark 2.5. From the above corollary, one can derive the numerical condition that K_S is numerically a divisor, as shown by the following:

Example 2.6. We consider the case $n=2$, i.e., the case where the type is $(2, q_1, p_2)$. Then $\mathrm{Zykel}(C_1)=(\underbrace{2, \cdots, 2}_{q_1-2}, p_2)$ and $\mathrm{Zykel}(D_1)=(\underbrace{2, \cdots, 2}_{p_2-2})$. Given $a_2=a$, $p_2=p$, we obtain $b_2=(p-2)(a-2)$ and $q_1=(p-2)(a-1)+1$. Therefore if K_S is numerically a divisor then the type is

$$(2, (p-2)(a-1)+1, p), \quad a\geqq 2, \quad p\geqq 3.$$

In this case, K_S can be written as

$$K_S\equiv -(a-1)B_1-2(a-1)B_2-\cdots-(p-1)(a-1)B_{p-1}$$
$$-((p-1)(a-1)-1)A_1-\cdots-aA_{N-p+1}$$

where $N=b_2(S)=(p-2)a$.

References

[1] G. Dloussky: Une propriété topologique des surfaces de Kato (preprint).

[2] I. Enoki: Surfaces of class VII_0 with curves. Tohoku Math. J., **33**, 453–492 (1981).

[3] Ma. Kato: Compact complex manifolds containing "global spherical shells", I. Proc. Int. Symp. Algebraic Geometry, Kyoto, Kinokuniya, Tokyo, pp. 45–84 (1977).

[4] I. Nakamura: On surfaces of class VII_0 with global spherical shells. Proc. Japan Acad., **59**A, 29–32 (1983).

[5] ——: On surfaces of class VII_0 with curves. Invent. Math., **78**, 393–443 (1984).

[6] K. Nishiguchi: Degeneration of surfaces with trivial canonical bundles. Proc. Japan Acad., **59**A, 304–307 (1983).

[7] ——: Degeneration of $K3$ surfaces (in preparation).