62. Proof of Masser's Conjecture on the Algebraic Independence of Values of Liouville Series

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Let $f(z)=\sum_{k=1}^{\infty} z^{k!}$. Then $f(z)$ converges in $|z|<1$. If α is an algebraic number with $0<|\alpha|<1$, then $f(\alpha)$ is a transcendental number. Masser conjectured that if $\alpha_1, \cdots, \alpha_n$ are algebraic numbers with $0<|\alpha_i|<1$ $(1\leqq i \leqq n)$ and no α_i/α_j $(1\leqq i<j\leqq n)$ is a root of unity, then $f(\alpha_1), \cdots, f(\alpha_n)$ are algebraically independent. In [2], the author proved the p-adic analogue of the conjecture, and in [3], settled the conjecture for $n=3$ in complex case. In this paper we shall prove the following theorem by using Evertse's Theorem 1 in [1].

Theorem. *Suppose $\alpha_1, \cdots, \alpha_n$ are algebraic numbers with $0<|\alpha_i|<1$ $(1\leqq i\leqq n)$ and no α_i/α_j $(1\leqq i<j\leqq n)$ is a root of unity. Then $f^{(l)}(\alpha_i)$ $(1\leqq i \leqq n,\ 0\leqq l)$ are algebraically independent, where $f^{(l)}(z)$ denotes the l-th derivative of $f(z)$.*

In what follows, K will denote an algebraic number field including $\alpha_1, \cdots, \alpha_n$. By a prime on K we mean an equivalence class of non-trivial valuations on K. We denote the set of all primes on K by S_K and the set of all infinite primes on K by S_∞. For every prime v on K lying above a prime p on $\boldsymbol{Q}$, we choose a valuation $\|\cdot\|_v$ such that

$$\|\alpha\|_v=|\alpha|_p^{[K_v:\,\boldsymbol{Q}_p]} \qquad \text{for all } \alpha\in \boldsymbol{Q}.$$

Then we have the product formula:

$$\prod_{v\in S_K} \|\alpha\|_v=1 \qquad \text{for all } \alpha\in K,\ \alpha\neq 0.$$

For $X=(x_0:x_1:\cdots:x_n)\in P^n(K)$, put

$$H_K(X)=H(X)=\prod_{v\in S_K} \max(\|x_0\|_v, \|x_1\|_v, \cdots, \|x_n\|_v).$$

By the product formula, this height is well-defined. Put

$$h_K(\alpha)=h(\alpha)=H(1:\alpha) \qquad \text{for } \alpha\in K.$$

Then so-called fundamental inequality holds,

$$-\log h(\alpha)\leqq \sum_{v\in S} \log\|\alpha\|_v \leqq \log h(\alpha) \qquad \text{for } \alpha\in K,\ \alpha\neq 0,$$

where S is any set of primes on K.

Let S be a finite set of primes on K, enclosing S_∞, and c, d be constants with $c>0$, $d\geqq 0$. A projective point $X\in P^n(K)$ is called (c, d, S)-*admissible* if its homogeneous coordinates $x_0, x_1, \cdots, x_n$ can be chosen such that

(i) all x_k are S-integers, i.e. $\|x_k\|_v\leqq 1$ if $v\notin S$

and

(ii) $\prod_{v\in S}\prod_{k=0}^{n} \|x_k\|_v\leqq c\cdot H(X)^d$.

The following theorem is due to Evertse [1]: *Let c, d be constants with $c>0$, $0\leqq d<1$ and let n be a positive integer. Then there are only finitely many (c, d, S)-admissible projective points $X=(x_0:x_1:\cdots:x_n)\in P^n(K)$ satisfying*

$$x_0+x_1+\cdots+x_n=0$$

but

$$x_{i_0}+x_{i_1}+\cdots+x_{i_s}\neq 0$$

for each proper, non-empty subset $\{i_0, i_1, \cdots, i_s\}$ of $\{0, 1, \cdots, n\}$.

Proof of Theorem. We may assume

$$|\alpha_1|=\cdots=|\alpha_t|>|\alpha_{t+1}|\geqq\cdots\geqq|\alpha_n|.$$

We prove the theorem by induction on n. If $n=0$, then the theorem is true. We suppose $n>0$ and $f^{(l)}(\alpha_i)$ $(1\leqq i\leqq n,\ 0\leqq l\leqq L)$ are algebraically dependent. Define $U\in \boldsymbol{C}^{n(L+1)}$ by

$$U=(\alpha_i^l f^{(l)}(\alpha_i))_{1\leqq i\leqq n, 0\leqq l\leqq L}.$$

Then there is a nonzero polynomial $F\in \boldsymbol{Z}[y_{10}, y_{11}, \cdots, y_{nL}]$ such that $F(U)=0$. We may assume F has the least total degree among them. By the assumption of induction, for any i, there exists a number l $(0\leqq l\leqq L)$ such that $\partial F/\partial y_{il}\neq 0$, and so $\partial F/\partial y_{il}(U)\neq 0$. Define $U_m\in \boldsymbol{C}^{n(L+1)}$ by

$$U_m=\left(\sum_{k=1}^{m-1} k!\,(k!-1)\cdots(k!-l+1)\alpha_i^{k!}\right)_{1\leqq i\leqq n, 0\leqq l\leqq L}.$$

Then $\lim_{m\to\infty} U_m=U$ and

$$-F(U_m)=F(U)-F(U_m)=\sum_{|J|\geqq 1} J!^{-1}\partial^{|J|}F/\partial y^J(U_m)(U-U_m)^J,$$

where $J=(j_{10}, j_{11}, \cdots, j_{nL})$ with j_{il} being non negative integers and $|J|$, $J!$, $\partial^{|J|}/\partial y^J$ and $(U-U_m)^J$ are defined in the usual way. Then

$$\begin{aligned}(1)\qquad -F(U_m)&=\sum_{i=1}^{t}\sum_{l=0}^{L}\partial F/\partial y_{il}(U_m)m!\,(m!-1)\cdots(m!-l+1)\alpha_i^{m!}\\ &\quad+O(m!^L|\alpha_{t+1}|^{m!})+O(m!^{2L}|\alpha_1|^{2m!})\\ &=O(m!^L|\alpha_1|^{m!}).\end{aligned}$$

On the other hand $h(F(U_m))\leqq c_1^{(m-1)!}$. Hence by the fundamental inequality, we have $F(U_m)=0$ for sufficiently large m. By (1),

$$(2)\qquad \sum_{i=1}^{t}\sum_{l=0}^{L}\partial F/\partial y_{il}(U_m)m!\,(m!-1)\cdots(m!-l+1)\alpha_i^{m!}=O(A^{m!}),$$

where $\max(|\alpha_1|^2, |\alpha_{t+1}|)<A<|\alpha_1|$. Put

$$\beta_i(m)=\sum_{l=0}^{L}\partial F/\partial y_{il}(U_m)m!\,(m!-1)\cdots(m!-l+1).$$

Then there is a positive number M such that $\beta_i(m)\neq 0$ $(1\leqq i\leqq t)$ for $m>M$, since there exists l $(0\leqq l\leqq L)$ such that $\partial F/\partial y_{il}(U)\neq 0$. We have

$$(3)\qquad \sum_{i=1}^{t}\beta_i(m)\alpha_i^{m!}=O(A^{m!})$$

and

$$(4)\qquad h(\beta_i(m))\leqq c_2^{(m-1)!}.$$

If $t=1$, (3) and (4) contradict each other, and the theorem is proved. In what follows, we assume $t>1$.

Proposition 1. *Let $\{i_1, \cdots, i_s\}$ be any subset of $\{1, \cdots, t\}$ with $s\geqq 2$*

and let $m_1 > m_2 > M$. *If* m_1 *is sufficiently large, then*

$$(\beta_{i_1}(m_1)\alpha_{i_1}^{m_1!} : \cdots : \beta_{i_s}(m_1)\alpha_{i_s}^{m_1!}) \neq (\beta_{i_1}(m_2)\alpha_{i_1}^{m_2!} : \cdots : \beta_{i_s}(m_2)\alpha_{i_s}^{m_2!}).$$

Proof. Suppose the proposition is not true. Then

$$\beta_{i_1}(m_1)\alpha_{i_1}^{m_1!}\beta_{i_2}(m_2)\alpha_{i_2}^{m_2!} = \beta_{i_2}(m_1)\alpha_{i_2}^{m_1!}\beta_{i_1}(m_2)\alpha_{i_1}^{m_2!},$$

and so

$$h(\alpha_{i_2}/\alpha_{i_1})^{m_1!-m_2!} \leqq c_2^{4(m_1-1)!}$$

for infinitely many m_1. Since $h(\alpha_{i_2}/\alpha_{i_1}) > 1$ and $m_1! - m_2! \geqq (m_1-1)(m_1-1)!$, this is a contradiction.

Proposition 2. *Let* $\{i_1, \cdots, i_s\}$ *be any non-empty subset of* $\{1, \cdots, t\}$. *Then*

$$\sum_{i \in \{i_1, \cdots, i_s\}} \beta_i(m)\alpha_i^{m!} \neq 0$$

for sufficiently large m.

Proof. Let S be a finite set of primes on K which includes S_∞ and all divisors of α_i $(1 \leqq i \leqq n)$. Then $\beta_i(m)\alpha_i^{m!}$ are S-integers. We prove the proposition by induction on s. If $s=1$, the proposition is true. We suppose $s \geqq 2$ and

$$\sum_{i \in \{i_1, \cdots, i_s\}} \beta_i(m)\alpha_i^{m!} = 0$$

for infinitely many m. Let ε be any positive number <1. By Evertse's theorem and Proposition 1,

$$\prod_{v \in S} \prod_{i \in \{i_1, \cdots, i_s\}} \|\beta_i(m)\alpha_i^{m!}\|_v > H(\beta_{i_1}(m)\alpha_{i_1}^{m!} : \cdots : \beta_{i_s}(m)\alpha_{i_s}^{m!})^{1-\varepsilon},$$

for infinitely many m. By the fact that $\prod_{v \in S} \|\alpha_i^{m!}\|_v = 1$ and there exists a prime v such that $\|\alpha_{i_2}/\alpha_{i_1}\|_v > 1$, we have

$$c_2^{s(m-1)!} > (\|\beta_{i_2}(m)/\beta_{i_1}(m)\|_v \|\alpha_{i_2}/\alpha_{i_1}\|_v^{m!})^{1-\varepsilon}.$$

This is a contradiction.

Now we complete the proof of the theorem. By the equality (3),

$$\sum_{i=1}^{t} \beta_i(m)\alpha_i^{m!} + \delta_m = 0, \text{ where } \delta_m = O(A^{m!}).$$

Let ε be any positive number <1. We may assume K is not a real field and $|\cdot|^2 = \|\cdot\|_{v_0}$ for some infinite prime v_0 on K. By Proposition 1, Proposition 2 and Evertse's theorem, we have

$$(5) \qquad \prod_{v \in S} \prod_{i=1}^{t} \|\beta_i(m)\alpha_i^{m!}\|_v \times \prod_{v \in S} \|\delta_m\|_v > H(\beta_1(m)\alpha_1^{m!} : \cdots : \beta_t(m)\alpha_t^{m!} : \delta_m)^{1-\varepsilon}$$

if m is sufficiently large. The left hand side of the inequality (5) is not greater than

$$c_3^{(m-1)!}\Big(\prod_{\substack{v \in S \\ v \neq v_0}} \max(\|\alpha_1\|_v, \cdots, \|\alpha_t\|_v)^{m!}\Big)A^{2m!}.$$

The right hand side of the inequality (5) is not less than

$$c_4^{(m-1)!} \prod_{v \in S} \max(\|\alpha_1\|_v, \cdots, \|\alpha_t\|_v)^{m!(1-\varepsilon)}.$$

Then we have

$$c_5^{(m-1)!}A^{2m!} \geqq |\alpha_1|^{2m!(1-\varepsilon)} \prod_{\substack{v \in S \\ v \neq v_0}} \max(\|\alpha_1\|_v, \cdots, \|\alpha_t\|_v)^{-\varepsilon m!}$$

for sufficiently large m. This contradicts the fact $A<|\alpha_1|$, and the theorem is proved.

References

[1] J.-H. Evertse: On sums of S-units and linear recurrences. Comp. Math., **53**, 225–244 (1984).
[2] K. Nishioka: Algebraic independence of certain power series of algebraic numbers (to appear in J. Number Theory).
[3] ——: Algebraic independence of three Liouville numbers (to appear in Arch. Math.).