39. On a Criterion for Hypoellipticity

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Introduction and main theorems. In this note we give a sufficient condition for second order differential operators to be hypoelliptic. The condition is also necessary for a special class of differential operators.

Let Ω be an open set in R^n and let $P=p(x, D_x)$ be a second order differential operator with real valued coefficients in $C^\infty(\Omega)$. Let (u, v) denote the inner product of u, v in L^2 and $\|u\|^2=(u, u)$. Let $\|\cdot\|_s$ denote the Sobolev space H_s for real s.

Theorem 1. *Assume that for any $\varepsilon>0$ and any compact set K of Ω there is a constant $C_{\varepsilon,K}$ such that*

$$(1)\qquad \|(\log\langle D_x\rangle)^2 u\| \leqq \varepsilon\|Pu\|+C_{\varepsilon,K}\|u\|, \qquad u\in C_0^\infty(K),$$

where $\log\langle D_x\rangle$ *denotes a pseudodifferential operator with a symbol* $\log\langle\xi\rangle$, $\langle\xi\rangle^2=|\xi|^2+1$. *Assume that the estimate*

$$(2)\qquad \sum_{j=1}^n (\|P^{(j)}u\|^2+\|P_{(j)}u\|_{-1}^2) \leqq C(\mathrm{Re}\,(Pu, u)+\|u\|^2), \qquad u\in C_0^\infty(K)$$

holds for a constant $C=C_K$, where $P^{(j)}=\partial_{\xi_j}p(x,\xi)$ and $P_{(j)}=D_{x_j}p(x,\xi)$. Then P is hypoelliptic in Ω. Furthermore we have $\mathrm{WF}\,Pu=\mathrm{WF}\,u$ *for* $u\in\mathcal{D}'(\Omega)$.

We remark that the hypothesis of (2) is removable if the principal symbol of P is non-negative. The estimate (1) is not always necessary for the hypoellipticity. We have a counter example $D_{x_1}^2+\exp(-1/|x_1|^\delta)D_{x_2}^2$ for $\delta\geqq 1$ given by [1] (cf. [6]). However, for a class of differential operators, the estimate (1) is necessary to be hypoelliptic. The result is extendible to operators of higher order. Let m be an even positive integer and let P_0 be a differential operator of the form

$$(3)\qquad P_0=D_t^m+\mathcal{A}(x, D_x) \qquad \text{in } R_t\times R_x^n,$$

where $\mathcal{A}(x, D_x)$ is a differential operator of order m with C^∞-coefficients and formally self-adjoint in an open set Ω of R_x^n. We assume that $\mathcal{A}(x, D_x)$ admits a positive self-adjoint realization $(A, D(A))$ in $L^2(\Omega)$.

Theorem 2. *Let P_0 be the operator defined above. Assume that P_0 is hypoelliptic in $R_t\times\Omega$. Then for any $(t_0, x_0)\in R_t\times\Omega$ one can find a neighborhood ω of x_0 satisfying the following: For any $\varepsilon>0$ there is a constant C_ε such that*

$$(4)\qquad \|(\log\langle D_t, D_x\rangle)^{m/2}u\|^2 \leqq \varepsilon\,\mathrm{Re}\,(P_0u, u)+C_\varepsilon\|u\|^2, \qquad u\in C_0^\infty(R_t\times\omega).$$

We remark that when $m=2$ the estimate (1) follows from (4) by means of the partition of unity over K and the replacement of u by $(\log\langle D_t, D_x\rangle)u$.

Our two theorems are applicable to the hypoellipticity for operators considered in [8] and [9]. Especially, an application shows that $D_t^2+D_{x_1}^2+\exp(-1/|x_1|^\delta)D_{x_2}^2$, $\delta>0$, is hypoelliptic in R^3 if and only if $\delta<1$ (cf. Theorem 8.41 of [4]). As another application we give:

Theorem 3. *Set* $P_1=D_t^2+x_2^2D_{x_1}^2+D_{x_2}^2+D_{x_3}(\sigma(x_1)\tau(x_3))D_{x_3}$, *where* $\sigma, \tau\in C^\infty$, $\tau>0$, $\sigma(0)=0$, $\sigma(s)>0$ $(s\neq 0)$ *and* $s\sigma'(s)\geqq 0$. *Then* P_1 *is hypoelliptic in* R^4 *if and only if* $\sigma(s)$ *satisfies*

$$\lim_{s\to 0}|s^{1/2}\log\sigma(s)|=0. \tag{5}$$

When τ is the constant the necessity of (5) can be also proved by the similar method as in [8].

1. Proof of Theorem 1. Let $h(x)\in C_0^\infty(R^n)$ be 1 for $|x|\leqq 1/2$ and vanish for $|x|\geqq 3/4$. Write $p(x,\xi)=\sum_{k=0}^2 p_k(x,\xi)$, where p_k is positively homogeneous in ξ of degree k. For $\gamma\equiv(x_0,\bar\xi_0)\in\Omega\times S^{n-1}$ we consider a differential operator

$$P_\gamma=p_\gamma(\lambda y,\lambda D_y)=\sum_{k=0}^2 p_k(x_0+\lambda y,\bar\xi_0+\lambda D_y)\lambda^{-2k} \tag{6}$$

with a small parameter $\lambda>0$ (see §3 of [2] and §2 of [7]). Substituting $u=h(x-x_0)h(\lambda^2D_x-\bar\xi_0)v(\lambda^{-1}(x-x_0))\exp(i\lambda^{-2}x\cdot\bar\xi_0)$, $v\in\mathcal{S}$, into (1) and (2) we have:

Lemma 1. *If* (1) *and* (2) *hold then for any real* $s>0$ *and any* $\gamma=(x_0,\bar\xi_0)\in\Omega\times S^{n-1}$ *there are a constant* $\lambda_0=\lambda_0(s,\gamma)$ *and a constant* C_γ *independent of* s *such that with* $H=h(\lambda D_y)h(\lambda y)$ *and* $H_0=h(\lambda D_y/2)h(\lambda y/2)$ *we have*

$$\begin{aligned}(\log\lambda^{-s})^2\|Hv\|+(\log\lambda^{-s})\sum_{j=1}^n(\|HP_\gamma^{(j)}v\|+\lambda^2\|HP_{(j)\gamma}v\|)\\ \leqq C_\gamma\|H_0P_\gamma v\|+C(s,\gamma)\|(1-H)v\|,\qquad v\in\mathcal{S},\end{aligned} \tag{7}$$

if $0<\lambda\leqq\lambda_0$, *where* $C(s,\gamma)$ *is a constant independent of* λ.

Set $h_\delta(x)=h(x/\delta)$ for a small $0<\delta\leqq 1/8$. Using (7) repeatedly we show that for reals s, s', $\kappa>0$ there is a constant $C=C(s,s')$ independent of κ such that

$$\|\Lambda_{k,\kappa}h_\delta(x-x_0)u\|_s\leqq C(\|\Lambda_{k,\kappa}h_{2\delta}(x-x_0)Pu\|_s+\|u\|_{-s'}),\qquad u\in C_0^\infty, \tag{8}$$

where $k=s+s'+2$ and $\Lambda_{k,\kappa}$ is a pseudodifferential operator with a symbol $(1+\kappa\langle\xi\rangle)^{-k}$. The detail of the proof will be given elsewhere.

2. Proof of Theorem 2. The method used here is only a version of the one in [5] p. 840–849, where non-analytic hypoellipticity for operators of the same form as (3) was studied (see Corollaries 3.6–7 of [5]). For the proof it suffices to derive the following estimate with $r=1/2$ (cf. (3.10) of [5])

$$\|(\log\langle D_x\rangle)^{mr}u\|^2\leqq\varepsilon\|A^ru\|^2+C_\varepsilon\|u\|^2,\qquad u\in C_0^\infty(\omega). \tag{9}$$

We may assume x_0 is the origin. We use the same notation as in [5]. Let $\psi\in C_0^\infty(\Omega)$ equal 1 in $\Pi=((-a,a))^n\Subset\Omega$. The hypothesis of the hypoellipticity implies that $u\in G^1(\Omega;\mathcal{A})\Rightarrow\psi u\in\mathcal{S}$ and hence $u\in D_\delta^1(A)\Rightarrow\psi u\in\mathcal{S}$ for a fixed $\delta>0$. The Banach closed graph theorem shows that for any integer $k>0$ there is a constant M_k such that

$$\sup_\xi|\langle\xi\rangle^{2k}\widehat{\psi u}(\xi)|\leqq M_k(N_\delta^1(u))^{1/2},\qquad u\in D_\delta^1(A). \tag{10}$$

In view of (3.4) of [5], it is clear that for any k there is a constant $M'_k \geqq 1$ such that

$$J^L_k(u) \leqq e^{2k} \|(L+1)^k u\|^2_{L^2(\Pi)} \leqq M'_k \|\langle\xi\rangle^{2k} \widehat{\psi u}\|^2, \tag{11}$$

where $J^L_k(u)$ denotes $J_k(u)$ defined from the spectrum resolution of L. Here $(L, D(L))$ is the realization of Legendre operator $\sum_{j=1}^n \partial_{x_j}(x_j^2 - a^2)\partial_{x_j}$ (see [5] p. 845). In what follows, to make clear the correspondence to A or L we often use the super script. Set $K_k = \{\xi ; \langle\xi\rangle \geqq M'_k M_{k+2}\}$. Then from (10) and (11) we have

$$\begin{aligned} J^L_k(u) &\leqq \|(M'_k M_{k+2}/\langle\xi\rangle) M^{-1}_{k+2} \langle\xi\rangle^{2k+2} \widehat{\psi u} \langle\xi\rangle^{-1}\|^2_{L^2(K_k)} \\ &\quad + M'_k \|\langle\xi\rangle^{2k} \widehat{\psi u}\|^2_{L^2(R^n_\xi \setminus K_k)} \\ &\leqq N^1_\delta(u) + C_k \|u\|^2_{L^2(\Omega)}, \qquad u \in D^1_\delta(A), \end{aligned} \tag{12}$$

with a constant C_k. Set $u(t) = F^A(t)u$. Then the estimate (12) and Lemma 3.1 of [5] show that for any $r > 0$ and $k > 0$

$$\begin{aligned} I_{r,k}(u(\cdot)) &\equiv \int_1^\infty \{\exp(-\delta(\mathrm{e}t)^{1/m}) J^L_k(u(t)) + \|u(t)\|^2_{L^2(\Pi)}\} t^{2r} \frac{dt}{t} \\ &\leqq 2 J^A_r(u) + C'_k \|u\|^2_{L^2(\Omega)}, \qquad u \in D(A^r) \end{aligned} \tag{13}$$

holds with a constant C'_k. We need replace Lemma 3.2 of [5] by

Lemma 2. *Let $t \to u(t)$ be a measurable mapping from $[1, \infty)$ to $L^2(\Pi)$ and let $I_{r,k}(u(\cdot))$ denote the integral defined by the formula* (13). *Assume that for reals $\delta > 0$, $r > 0$ and an integer $k > 0$ the integral $I_{r,k}(u(\cdot))$ is bounded. Then the integral $u = \int_1^\infty u(t)(dt/t)$ is convergent, $u \in D((\log(L+1))^{mr})$ and for a constant C independent of k we have*

$$k^{2mr} \|(\log(L+1))^{mr} u\|^2_{L^2(\Pi)} \leqq C I_{r,k}(u(\cdot)). \tag{14}$$

The proof of the lemma is parallel if we set $\sigma(t, \lambda) = \exp(2k \log \lambda - \delta \mathrm{e}^{1/m} t^{1/m})$ and $t(\lambda) = \mathrm{e}^{-1}((k/\delta)\log\lambda)^m$. We note that

$$\|(\log(L+1))^r u\|^2_{L^2(\Pi)} \leqq \int_1^\infty (\log\lambda)^{2r} \|F^L(\lambda)u\|^2_{L^2(\Pi)} \frac{d\lambda}{\lambda}$$

holds similarly to (3.4) of [5]. Set $\omega = ((-a/2, a/2))^n$. Then there is a constant C such that

$$\|(\log\langle D_x\rangle)^{mr} u\|^2 \leqq C(\|(\log(L+1))^{mr} u\|^2 + \|u\|^2), \qquad u \in C_0^\infty(\omega), \tag{15}$$

because we have $(\log(L+1))^{mr} = (L+1)(L+1)^{-1}(\log(L+1))^{mr}$ and, in ω, $(L+1)^{-1}(\log(L+1))^{mr}$ equals a pseudodifferential operator modulo smoothing operator with principal symbol $(l+1)^{-1}(\log(l+1))^{mr}$, $l = l(x, \xi) = \sum_{j=1}^n (a^2 - x_j^2)\xi_j^2$ (cf. Chapter 8 of [3]). Since we can take any large k, from (13)–(15) we obtain (9).

3. Proof of necessity of (5). In view of the proof of Theorem 2 we may use (9) instead of (4). We employ the localized form of (9) with $r = 1$ as follows: for $0 < \lambda \leqq 1$

$$\begin{aligned} (\log\lambda^{-1})^4 \|v\|^2 &\leqq \varepsilon \|A_\gamma v\|^2 + C_\varepsilon(\|v\|^2 \\ &\quad + \lambda^{-8}(\sum_{|\alpha| \leqq 4} \|\exp(-1/\lambda|y|)(\lambda D_y)^\alpha v\|^2 \\ &\quad + \sum_{|\alpha| = 4} \|(\lambda D_y)^\alpha v\|^2), \qquad v \in C_0^\infty, \end{aligned} \tag{16}$$

where A_γ is defined from $\mathcal{A}(x, D_x)$ by the same way as for P_γ. Set $\gamma=(0, \bar{\xi}_0)$, $\bar{\xi}_0=(0, 0, 1)$. Take a change of variables $\lambda y_1=\kappa(\log \lambda^{-1})^{-2}\tilde{y}_1$, $\lambda y_2=\kappa(\log \lambda^{-1})^{-1}\tilde{y}_2$, $y_3=\tilde{y}_3$, where $\kappa>0$ is a small parameter. Then the estimate (16) after the change of variables shows the necessity of (5) by means of the reductive absurdity.

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