12. Equilibrium Measures on Recurrent Markov Processes

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1. **Introduction.** We consider the potential theory for recurrent Markov processes introduced by T. Ueno [4]. He studied a pair of measures μ_L^K and μ_K^L satisfying $\mu_L^K(\cdot)=\mu_K^L h_K(\cdot)$, $\mu_K^L(\cdot)=\mu_L^K h_L(\cdot)$, where $h_K(x,\cdot)$ is the hitting measure to the set K. In this paper we prove that in the symmetric case the measure ν_L^K multiplied μ_L^K by the Ueno capacity is the equilibrium measure on $K\subset L^c$. Further we show that the equilibrium potential induced by ν_L^K is the hitting probability for K before attaining to L. We anticipate that such a pair of measures μ_L^K and μ_K^L is a new probabilistic characterization of the equilibrium measure.

2. **Preliminaries.** We refer the reader to [2] for all terminology and notation not explicitly defined here. Let R be a separable Hausdorff locally compact space containing at least two points and satisfying

(R.1) For each point $x\in R$, we can take a countable base of neighborhoods of x consisting of arcwise connected open sets,

(R.2) R is connected.

We denote by $\boldsymbol{B}$ the topological Borel field of subsets of R. For a set $A\in\boldsymbol{B}$ and a path function $X(t)$ from $[0,\infty)$ to R, σ_A is defined by

$$\begin{aligned}\sigma_A&=\inf\{t\geqq 0\mid X(t)\in A\}, &&\text{if such } t \text{ exists,}\\ &=\infty, &&\text{otherwise.}\end{aligned}$$

We denote by $\mathscr{B}$, the smallest Borel field of subsets of the sample space W containing $\{w\mid X(t,w)\in A\}$ for all $A\in\boldsymbol{B}$ and $t\geqq 0$. Let $\{P_x(\cdot), x\in R\}$ be a system of probability measures on satisfying

(P.1) $P_x(E)$ is a $\boldsymbol{B}$-measurable function of x for each $E\in\mathscr{B}$,

(P.2) $P_x(\{w\mid X(0,w)=x\})=1$ for each $x\in R$,

(P.3) quasi-left continuity,

(P.4) Markov property.

In order to study a broad class of recurrent Markov process Ueno [4] introduced the following assumptions (X.1)$\sim$(X.5) which we follow.

(X.1) Recurrence: $P_x(X(t)\in A$ for some $0\leqq t<\infty)=1$ for any $x\in A$, $A\in\boldsymbol{B}$.

We define the hitting measure $h_A(x,\cdot)$ for the set $A\in\boldsymbol{B}$ by

$$h_A(x,E)=P_x(X(\sigma_A)\in E,\ \sigma_A<\infty),\quad x\in R,\quad E\in\boldsymbol{B}.$$

(X.2) For any continuous function f on A,

$$h_A f(x)=\int h_A(x,dy)f(y)$$

is continuous in A^c, where A is a closed set in R containing an inner point.

(X.3) For non-negative continuous function f in A, $h_A f(x)$ is either strictly positive or 0 for all points x of any one component of A^c, where A is a closed set in R containing an inner point.

(X.4) For any continuous function f on R, the resolvent operator is continuous on R.

(X.5) There is no point of positive holding time.

Now, we introduce the Green measure

$$G_L(x, A)=E_x\left(\int_0^{\sigma_L} \chi_A(X(t))\,dt\right), \quad x \in R, \quad A \in \boldsymbol{B},$$

for any closed set L containing an inner point, where χ_A takes 1 on A, 0 on A^c respectively. Let $\mathcal{F}$ be the family of all $\{K, L\}$, where K and L are mutually disjoint closed sets in R and in particular K is compact. Ueno [4] proves that for each $\{K, L\} \in \mathcal{F}$ there is a unique pair of measures μ_L^K and μ_K^L with total mass 1 on K and L respectively, satisfying

$$\mu_L^K(\cdot)=\mu_K^L h_K(\cdot)=\int_L \mu_K^L(dx) h_K(x, \cdot),$$
$$\mu_K^L(\cdot)=\mu_L^K h_L(\cdot)=\int_K \mu_L^K(dx) h_L(x, \cdot).$$

Applying these μ_L^K, μ_K^L, Ueno introduces his own Green capacity. For $\{K, L\}$ and $\{K', L'\}$ in $\mathcal{F}$, put

(2.1) $$C_{(K,L)}(K', L)=\mu_L^K h_{K',L}(K'),$$
$$C_{(K',L)}(K, L)=C_{(K,L)}(K', L)^{-1}, \qquad \text{when } K' \subset K,$$

where

(2.2) $$h_{K,L}(x, E)=P_x(\sigma_K<\sigma_L,\ X(\sigma_K) \in E), \qquad E \in \boldsymbol{B}.$$
$$C_{(K,L)}(K', L)=C_{(K,L)}(K \cup K', L)\cdot C_{(K\cup K',L)}(K', L),$$

when $\{K, L\} \leftrightarrow \{K', L\}$, where the notation $\{K, L\} \leftrightarrow \{K', L\}$ denotes $\{K \cup K', L\} \in \mathcal{F}$. For a sequence $\alpha=(\{K_1, L_1\}, \{K_2, L_2\}, \cdots, \{K_n, L_n\})$ of satisfying $\{K, L\} \leftrightarrow \{K_1, L_1\} \leftrightarrow \cdots \leftrightarrow \{K_n, L_n\} \leftrightarrow \{K', L'\}$

(2.3) $$C_{(K,L)}^{\alpha}(K', L')=C_{(K,L)}(K_1, L_1)\cdot C_{(K_1,L_1)}(K_2, L_2)\cdot \cdots \cdot C_{(K_n,L_n)}(K', L').$$

Lemma 3.2 in [4] shows that such $C_{(K,L)}^{\alpha}(K', L')$ does not depend on the choice of α. Now fixing any $\{K_0, L_0\} \in \mathcal{F}$, we call $C(K, L)=C_{(K_0,L_0)}(K, L)$ the Green capacity of K with respect to L. Setting

(2.4) $$\nu_L^K(\cdot)=C(K, L)\mu_L^K(\cdot),$$

we introduce the measure

(2.5) $$m(\cdot)=\int_K \nu_L^K(dx) G_L(x, \cdot)+\int_L \nu_K^L(dx) G_K(x, \cdot).$$

Then every Green measure $G_L(x, \cdot)$ is absolutely continuous relative to m, that is, it has a density function $g_L(x, y)$ satisfying

(2.6) $$G_L(x, A)=\int_A g_L(x, y) m(dy).$$

3. Theorems. In this section we add following assumptions regarding the density function of the Green measure.

(A.1) $g_L(x, y)$ is lower semi-continuous with respect to x.

(A.2) symmetry: $g_L(x, y)=g_L(y, x)$

holds almost everywhere relative to m.

Theorem 2 has no bearing on above both assumptions. Theorem 1 and Theorem 3 are regardless of the assumption (A.1).

Theorem 1. *Assume that* (A.2) *holds. For* $\{K, L\} \in \mathfrak{P}$ *we have*

$$g_L \nu_L^K = 1, \qquad \text{a.e. } (m) \text{ on } K.$$

Proof. Let E be any compact subset of K. Observe that $G_K(x, E) = 0$ for $x \in L$. Applying (2.5) and the symmetry (A.2), we get

$$\begin{aligned}\int_E m(dx) &= \int \nu_L^K(dx) G_L(x, E) + \int \nu_K^L(dx) G_K(x, E) \\ &= \int \nu_L^K(dx)\Big(\int_E g_L(x, y)\, m(dy)\Big) = \int \nu_L^K(dx)\Big(\int_E g_L(y, x)\, m(dy)\Big) \\ &= \int_E \Big(\int g_L(y, x) \nu_L^K(dx)\Big) m(dy) = \int_E g_L \nu_L^K(y) m(dy).\end{aligned}$$

This implies

$$g_L \nu_L^K = 1, \qquad \text{a.e. } (m) \text{ on } K.$$

By Theorem 1 ν_L^K is the equilibrium measure for the kernel $g_L(x, y)$. Moreover in virtue of (2.4)

$$\nu_L^K(R) = C(K, L) \mu_L^K(R) = C(K, L).$$

That is, the total measure of ν_L^K is equal to the Ueno capacity.

In the next place we show the important properties of ν_L^K under the general condition. Such properties are known in the Brownian case. See Port-Stone ([3], p. 191).

Theorem 2. *Suppose that* $\{K', L\}$, $\{K, L\} \in \mathfrak{P}$ *and* $K' \subset K$. *Then we have*

(i) $\nu_L^{K'} = \nu_L^K h_{K', L}$,

where $h_{K', L}$ *is defined by* (2.2),

(ii) $C(K', L) = \int \nu_L^K(dx) P_x(\sigma_{K'} < \sigma_L)$.

Proof. The first equality follows from Theorem 3.1 of Ueno [4]. By applying (2.1), (2.3) and the definition (2.4) of ν_L^K, we obtain

$$\begin{aligned}C(K', L) &= C(K, L) C_{(K, L)}(K', L) = C(K, L) \mu_L^K h_{K', L}(K') \\ &= \nu_L^K h_{K', L}(K') = \int \nu_L^K(dx) P_x(\sigma_{K'} < \sigma_L).\end{aligned}$$

Subsequently, we prove that the potential of ν_L^K is the hitting probability of K before reaching L.

Theorem 3. *Let* $g_L(x, y)$ *be symmetric. Assume that* μ_L^K *and for* $x \in (L \cup K)^c$, $h_{K, L}(x, \cdot)$ *are absolutely continuous with respect to the measure* m. *Then we have*

$$g_L \nu_L^K = P_\cdot(\sigma_K < \sigma_L), \qquad \text{a.e. } (m).$$

Proof. By the strong Markov property and (2.6), what is called the fundamental identity

$$g_L(x, y) = g_{L \cup K}(x, y) + \int h_{K, L}(x, dz) g_L(z, y) \tag{3.1}$$

is obtained almost everywhere in y relative to the measure m. According to (3.1) and the absolute continuity of ν_L^K we get for $x \in L^c$

$$g_L \nu_L^K(x) = g_{L \cup K} \nu_L^K(x) + \int h_{K, L}(x, dz) g_L \nu_L^K(z). \tag{3.2}$$

Now the first term of the right hand in (3.2) vanishes excepting m-measure zero on $(K\cup L)^c$. In fact for any compact set E contained in $(K\cup L)^c$

$$\int_E g_{L\cup K}\nu_L^K(x)m(dx)=\int G_{L\cup K}(x,E)\nu_L^K(dx)$$

and note that $G_{L\cup K}(x,E)=0$ when $x\in K$. Therefore it follows from (3.2) that

$$g_L\nu_L^K(x)=\int h_{K,L}(x,dz)g_L\nu_L^K(z),\qquad \text{a.e. } (m) \text{ on } (K\cup L)^c.$$

On the strength of Theorem 1 and the absolute continuity of $h_{K,L}(x,\cdot)$ we see that

$$g_L\nu_L^K=P\cdot(\sigma_K<\sigma_L),\qquad \text{a.e. } (m) \text{ on } (K\cup L)^c. \tag{3.3}$$

Moreover combining Theorem 1 with $P\cdot(\sigma_K<\sigma_L)=1$ on K, we find that the equality (3.3) holds almost everywhere on K. Therefore we have

$$g_L\nu_L^K=P\cdot(\sigma_K<\sigma_L),\qquad \text{a.e. } (m) \text{ on } L^c. \tag{3.4}$$

Also we can show that

$$g_L\nu_L^K=0,\qquad \text{a.e. } (m) \text{ on } L, \tag{3.5}$$

by the same method as the first term of the right hand in (3.2). We complete the proof of Theorem 3 by (3.4) and (3.5).

Proposition 1. *If f and g are excessive and $f=g$ except on a null set, then $f=g$ everywhere* (Blumenthal-Getoor [1], p. 80).

Proposition 2. *If for a subset A of R, τ_A is defined as*

$$\tau_A=\inf\{t>0\,|\,X(t)\in A\},\quad \textit{if such } t \textit{ exists},$$
$$=\infty,\quad \textit{otherwise},$$

then

(i) $\sigma_A\leqq\tau_A$ *and* $\sigma_A=\tau_A$ *if* $X(0)\notin A$,

(ii) $t+\sigma_A\circ\theta_t$ *is an increasing function of* t *and*

$$\lim_{t\downarrow 0}(t+\sigma_A\circ\theta_t)=\tau_A,$$

where θ_t denotes the shift transformation (Blumenthal-Getoor [1], p. 53).

Theorem 4. *Suppose all the assumptions in the previous theorem. If $g_L(x,y)$ is lower semi-continuous with respect to x, then we have*

$$g_L\nu_L^K=P\cdot(\sigma_K<\sigma_L),\qquad \text{on } L^c.$$

Proof. It suffices to prove that $g_L\nu_L^K$ and $P\cdot(\sigma_K<\sigma_L)$ are excessive on L^c. In order to consider the case of the potential $g_L\nu_L^K$, note that there exists a Borel function f such that $\nu_L^K(dx)=f(x)m(dx)$. Then we have

$$g_L\nu_L^K(x)=\int G_L(x,dy)f(y).$$

By Lemma 4.1 of Ueno [4] G_Lf is superharmonic on L^c, namely for every $x\in L^c$ and every open ball $V\subset L^c$ with the center x

$$g_L\nu_L^K(x)\geqq\int_{V^c} h_{V^c}(x,dy)g_L\nu_L^K(y).$$

By combining the assumption (A.1) with Fatou's lemma, $g_L\nu_L^K$ is lower semi-continuous on L^c. Hence $g_L\nu_L^K$ is excessive on L^c.

Next we show that $P\cdot(\sigma_K<\sigma_L)$ is excessive on L^c. Let

$$Q^t(x,E)=P_x(X(t)\in E,\ \sigma_L>t)$$

for $x \in L^c$, $t \geqq 0$ and $E \subset L^c$. Then it is sufficient to see that

(3.6) $$Q^t P_x(\sigma_K < \sigma_L) \leqq P_x(\sigma_K < \sigma_L),$$

and

(3.7) $$\lim_{t \downarrow 0} Q^t P_x(\sigma_K < \sigma_L) = P_x(\sigma_K < \sigma_L).$$

It follows from the Markov property that for $x \in L^c$ and $t > 0$

(3.8) $$\begin{aligned} Q^t P_x(\sigma_K < \sigma_L) &= \int P_x(X(t) \in dy,\ \sigma_L > t) P_y(\sigma_K < \sigma_L) \\ &= E_x(P_{X(t)}(\sigma_K < \sigma_L)\ ;\ \sigma_L > t) \\ &= P_x(\sigma_K \circ \theta_t < \sigma_L \circ \theta_t,\ \sigma_L > t) \\ &= P_x(t + \sigma_K \circ \theta_t < \sigma_L). \end{aligned}$$

By means of Proposition 2, $t + \sigma_K \circ \theta_t$ is monotonely increasing with respect to t and

$$t + \sigma_K \circ \theta_t \geqq \lim_{t \downarrow 0} (t + \sigma_K \circ \theta_t) = \tau_K \geqq \sigma_K.$$

Consequently the inequality (3.6) is shown for $t > 0$ with the help of (3.8). Since $Q^t P_x(\sigma_K < \sigma_L) = P_x(\sigma_K < \sigma_L)$ for $t = 0$, (3.6) is obvious. To check the relation (3.7) for $x \in L^c$, let t approach to 0 in (3.8).

(3.9) $$\lim_{t \downarrow 0} Q^t P_x(\sigma_K < \sigma_L) = \lim_{t \downarrow 0} P_x(t + \sigma_K \circ \theta_t < \sigma_L).$$

By applying (3.9) and Proposition 2, we have for $x \in (K \cup L)^c$

$$\lim_{t \downarrow 0} Q^t P_x(\sigma_K < \sigma_L) = P_x(\tau_K < \sigma_L) = P_x(\sigma_K < \sigma_L).$$

Also for $x \in K$, $\sigma_K = 0$. Thus in virtue of (3.9)

$$\lim_{t \downarrow 0} Q^t P_x(\sigma_K < \sigma_L) = \lim_{t \downarrow 0} P_x(t < \sigma_L) = P_x(0 < \sigma_L) = P_x(\sigma_K < \sigma_L).$$

Hence the equality (3.7) holds on L^c.

References

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