10. *Commutators on Dyadic Martingales*

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§1. **Introduction.** A characterization of $BMO(\boldsymbol{R}^n)$ by commutators with singular integrals was given by Coifman-Rochberg-Weiss [7]. (See also [8].) Later, an analogue for regular martingales is shown by Janson [9]. Recently, Chanillo [3] and Rochberg-Weiss [11] and Komori [10] obtained a similar result on commutators with fractional integrals. It is the purpose of this note to study fractional integrals and commutators in the dyadic martingale setting. A version of fractional integrals I^α for dyadic martingales is introduced which is parallel to that on Walsh-Fourier series studied by Watari [14], and that on local fields by Taibleson [13]. The boundedness of commutators $[b, I^\alpha]$ shall be used to characterize the multiplicating function b.

§2. **Fractional integrals.** Let $\mathcal{F}_n$ be the sub-σ- field generated by dyadic intervals of length 2^{-n} in $[0,1]$, $n=0,1,2,\cdots$. A martingale $\{f_n\}_{n\geq 0}$ relative to $\{\mathcal{F}_n\}_{n\geq 0}$ is a dyadic martingale. For an integrable function f on $[0,1)$, the conditional expectations $f_n\equiv E(f|\mathcal{F}_n)$, $n=0,1,2,\cdots$, form a dyadic martingale whose L^p norm, $\sup_n \|f_n\|_p$, equals to the L^p norm of the function f, for $p\geq 1$. We shall identify f with $\{f_n\}$ by writing $f=\{f_n\}$ and assume $f_0=0$. Let $\{d_n\}$ be the difference sequence of $f=\{f_n\}$, i.e. $f_n=\sum_{k=1}^n d_k$. The maximal function and square function of $f=\{f_n\}$ are given by $f^*=\sup|f_n|$ and $S(f)=(\sum_{k=1}^\infty d_k^2)^{1/2}$, respectively. The following are well-known. (See [1], [2] and [5].)

$$(1)\qquad \begin{array}{ll} \|f^*\|_p\approx\|f\|_p\approx\|S(f)\|_p, & \text{for } 1<p<\infty, \text{ and} \\ \|f^*\|_p\approx\|S(f)\|_p, & \text{for } 0<p<\infty. \end{array}$$

Now for a dyadic martingale $f=\{f_n\}$ and $\alpha\in\boldsymbol{R}$, we define the fractional integral $I^\alpha f=\{(I^\alpha f)_n\}$ of f (of order α) by $(I^\alpha f)_n=\sum_{k=1}^n 2^{-k\alpha}d_k$, whose maximal function is $(I^\alpha f)^*=\sup_n|\sum_{k=1}^n 2^{-k\alpha}d_k|$. If $\alpha>0$, $I^\alpha f$ is simply a martingale transform introduced by Burkholder [1]. It is trivial that $\|(I^\alpha f)^*\|_p \leq C\|I^\alpha f\|_p\leq C\|f\|_p$ for $0<\alpha<\infty$ and $1<p<\infty$. Moreover, we have

Theorem 1. *For integrable f,*

$$(2)\qquad \|(I^\alpha f)^*\|_q\leq C\|f\|_p \qquad \textit{where } 1<p<q<\infty$$

and $\alpha=1/p-1/q$;

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$$(3) \qquad P[(I^\alpha f)^* > \lambda] \le C\left(\frac{\|f\|_1}{\lambda}\right)^{1/(1-\alpha)} \qquad \textit{for all } \lambda > 0.$$

Note that Watari [14] proved these results for $|I^\alpha f|$ in the place of $(I^\alpha f)^*$ by using some orthogonal properties of the Walsh-Fourier series. (2) follows from his version and (1). (3) (as well as (2)) can be obtained by a Calderón-Zygmund type of decomposition argument (or a stopping time) for regular martingales similar to the one used in [4] and [6]. Another proof of (2) is by applying (1) to certain norm estimates of d_k.

§3. Commutators and BMO. Martingales of bounded mean oscillation (BMO) are those martingales $b=\{b_n\}$ such that

$$\sup_n \|E(|b-b_n||\mathcal{F}_n)\|_\infty \equiv \|b\|_* < \infty.$$

This is equivalent, for dyadic martingales, to that $\sup_n \|E(|b-b_{n-1}||\mathcal{F}_n)\|_\infty < \infty$. The John-Nirenberg inequality gives other equivalent norms:

$$\|b\|_* \approx \sup_n \|[E(|b-b_n|^s|\mathcal{F}_n)]^{1/s}\|_\infty \qquad \text{for each } 1 \le s < \infty.$$

The sharp function of b is given by $b^\# = \sup_n E(|b-b_n||\mathcal{F}_n)$. We note that $\|b^\#\|_\infty = \|b\|_*$ and $\|b^\#\|_p \approx \|b\|_p$, $1<p<\infty$.

For an integrable function b, we define the commutator with I^α by $[b, I^\alpha]f = bI^\alpha f - I^\alpha(bf)$.

Our main result generalizing the one in Euclidean spaces by Chanillo [3], Rochberg-Weiss [11] and Komori [10] is the following

Theorem 2. *Let $1<p<q<\infty$ and $\alpha = 1/p-1/q>0$. Then b is in BMO if and only if the commutator $[b, I^\alpha]$ is bounded from L^p to L^q*, i.e., $[b, I^\alpha] \in \boldsymbol{B}(L^p, L^q)$.

We need a preliminary result on $f^*_\alpha \equiv \sup_n 2^{-n\alpha}|f_n|$ where $f=\{f_n\}$ is a dyadic martingale.

Lemma. *Let $1<p<q<\infty$ and $\alpha=1/p-1/q$. Then $\|f^*_\alpha\|_q \le C\|f\|_p$.*

The lemma follows from a decomposition (or a stopping time) argument as mentioned before. See also [12] or [3].

Proof of Theorem 2. Suppose $b \in BMO$. Given a dyadic interval J in $\mathcal{F}_n$ with length 2^{-n}, let b_J be the average value of b on J. Write

$$\begin{aligned} g \equiv [b, I^\alpha]f &= (b-b_J)I^\alpha f - I^\alpha((b-b_J)f\chi_J) - I^\alpha((b-b_J)f\chi_{J^c}) \\ &\equiv g^{(1)} + g^{(2)} + g^{(3)}, \quad \text{say.} \end{aligned}$$

Now, we choose a t such that $1<t<q$ and $1/s+1/t=1$, then we have

$$\begin{aligned} E(|g^{(1)}||\mathcal{F}_n)(x) &= E(|(b-b_J)I^\alpha f||\mathcal{F}_n)(x) \\ &\le [E(|b-b_J|^s|\mathcal{F}_n)(x)]^{1/s}[E(|I^\alpha f|^t|\mathcal{F}_n)(x)]^{1/t} \\ &\le C_1\|b\|_*[(|I^\alpha f|^t)^*(x)]^{1/t}. \end{aligned}$$

To estimate $g^{(2)}$, we first choose p_1 and v such that $1<p_1<v<p$ and suppose $\alpha = 1/p_1 - 1/q_1$, $1/u+1/v=1/p_1$. We have $1<p_1<q_1<\infty$ and $1<u$, $v<\infty$. Then it follows from Theorem 1 that

$$\begin{aligned} E(|g^{(2)}||\mathcal{F}_n)(x) &\le [E(|I^\alpha((b-b_J)f\chi_J)|^{q_1}|\mathcal{F}_n)(x)]^{1/q_1} \\ &\le C_2 2^{-n\alpha}[E(|(b-b_J)f|^{p_1}|\mathcal{F}_n)(x)]^{1/p_1} \\ &\le C_2 2^{-n\alpha}[E(|b-b_J|^u|\mathcal{F}_n(x)]^{1/u}[E(|f|^v|\mathcal{F}_n)(x)]^{1/v} \\ &\le C_3\|b\|_*[(|f|^v)^*_{\alpha v}(x)]^{1/v}. \end{aligned}$$

Note that $g^{(3)}$ is constant on J. Hence we have

$$g^{\sharp}(x)\leq C\,\|b\|_{*}\,\{[(|I^{\alpha}f|^{t})^{*}(x)]^{1/t}+[(|f|^{v})^{*}_{av}(x)]^{1/v}\}.$$

Therefore, by Theorem 1 and Lemma, we obtain

$$\|[b,I^{\alpha}]f\|_{q}=\|g\|_{q}\leq C\,\|b\|_{*}\,\|f\|_{p}.$$

Conversely, consider a dyadic interval $J\in\mathcal{F}_{n}$ with length 2^{-n}. Let J_1 be its adjacent dyadic interval of the same size, i.e. $J\cup J_1\in\mathcal{F}_{n-1}$. An easy computation shows that for $x\in J$,

$$[b,I^{\alpha}]\chi_{J_1}(x)=(b(x)-b_{J_1})2^{-n\alpha}(2^{\alpha}-1-2^{-n(1-\alpha)})(2-2^{\alpha})^{-1}.$$

Hence if $n>N(\alpha)\equiv(\alpha-1)^{-1}\log_2(2^{\alpha}-1)$, then

$$|[b,I^{\alpha}]\chi_{J_1}(x)|\geq C(\alpha)2^{-n\alpha}\,|b(x)-b_{J_1}|,$$

for some $C(\alpha)>0$. Thus for $x\in J\in\mathcal{F}_{n}$ with $n>N(\alpha)$,

$$\begin{aligned}[E(|b-b_{J_1}|^{q}\,|\,\mathcal{F}_{n})(x)]^{1/q}&\leq C2^{n\alpha}[E(|[b,I^{\alpha}]\chi_{J_1}|^{q}\,|\,\mathcal{F}_{n})(x)]^{1/q}\\&\leq C2^{n\alpha}2^{n/q}\,\|[b,I^{\alpha}]\chi_{J_1}\|_{q}\\&\leq C_1 2^{n\alpha}2^{n/q}\,\|\chi_{J_1}\|_{p}=C_1\end{aligned}$$

where $C_1=\|[b,I^{\alpha}]\|/C(\alpha)$. This implies that $b\in BMO$.

Therefore the proof of Theorem 2 completed.

§4. Hardy and Lipschitz spaces. H^{p} martingales, $0<p<\infty$, are those martingales f whose maximal function f^{*} is in L^{p}. For $\lambda\in\boldsymbol{R}$, a dyadic martingale $f=\{f_n\}$ is said to be in Lip λ if

$$\|f\|_{(\lambda)}=\sup 2^{n\lambda}\,\|E(|f-f_n|\,|\,\mathcal{F}_{n})\|_{\infty}<\infty.$$

Note that Lip $0=BMO$ and for $0<p\leq 1$, the dual of H^{p} is Lip $(1/p-1)$.

The results in the previous sections about fractional integrals and commutators on L^{p} and BMO can be extended to H^{p} and Lipschitz spaces also. We shall state some generalizations and omit the proofs.

Theorem 3.

(i) $I^{\alpha}\in\boldsymbol{B}(H^{p},H^{q})$, $0<p<q<\infty$ *and* $\alpha=1/p-1/q$.

(ii) $I^{\alpha}\in\boldsymbol{B}(\mathrm{Lip}\,\lambda,\ \mathrm{Lip}\,(\alpha+\lambda))$, $0<\alpha,\ \lambda<\infty$.

(iii) $I^{\alpha}\in\boldsymbol{B}(BMO,\ \mathrm{Lip}\,\alpha)$, $\alpha>0$.

(iv) $I^{\alpha}\in\boldsymbol{B}(H^{p},\ \mathrm{Lip}\,(\alpha-1/p))$, $1<p<\infty$, $\alpha>1/p$.

(v) $I^{\alpha}\in\boldsymbol{B}(H^{p},BMO)$, $1<p<\infty$, $\alpha=1/p$.

Theorem 4. *Let* $1<p<q<\infty$, $\alpha+\lambda=1/p-1/q$ *and* $0<\alpha$, $\lambda<\infty$. *Then* $b\in\mathrm{Lip}\,\lambda$ *if and only if* $[b,I^{\alpha}]\in\boldsymbol{B}(L^{p},L^{q})$.

Finally, we remark that the results in this note can be easily generalized to regular martingales and to the local field setting.

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