91. Jacobian Rings of Hypersurfaces of Compact Irreducible Hermitian Symmetric Spaces and Generic Torelli Theorem

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0. The purpose of the present notes is to show some properties of Jacobian rings of smooth hypersurfaces of compact irreducible Hermitian symmetric spaces, in particular, the duality properties and the symmetrizer lemma. Using these properties, we shall prove the generic Torelli theorem for smooth hypersurfaces with the ample canonical bundles in such Hermitian symmetric spaces with a few exceptions (Theorem 4.1). In case of the projective space $\boldsymbol{P}^n$, the Jacobian ring of a smooth hypersurface is an *Artinian Gorenstein ring* by the local duality theorem (cf. [5]). Using this property, Donagi proved the *symmetrizer lemma* for projective hypersurfaces, which is a key step in his proof of the generic Torelli theorem for projective hypersurfaces (cf. [2], see also [8] for the weighted projective case). Contrary to the projective case, the Jacobian rings here are *not* Gorenstein in general and we shall give a sufficient condition for the duality property (DP) (Theorem 1.2, Theorem 2.1). By using this, we shall show the symmetrizer lemma, which is sufficient to prove our generic Torelli theorem. Details will be published elsewhere.

1. Let G be a complex simple Lie group and let U be a parabolic Lie subgroup such that the quotient manifold $Y=G/U$ is a compact irreducible Hermitian symmetric space. These Hermitian symmetric spaces are divided into six classes; I. $Gr(m+n, n)=U(m+n)/U(m)\times U(n)$, II. $SO(2n)/U(n)$, III. $Sp(n)/U(n)$, IV. $Q^n=SO(n+2)/SO(2)\times SO(n)$ $(n>2)$, V. $E_6/Spin(10)\times T^1$, VI. $E_7/E_6\times T^1$, (see, e.g., [1]). Let $H=\mathcal{O}_Y(1)$ denote the ample generator of the Picard group $\mathrm{Pic}(Y)\cong \boldsymbol{Z}$ and we shall write $\mathcal{O}_Y(a)$ instead of $H^{\otimes a}$.

Let $X\subset Y$ be a *smooth hypersurface of degree d*, defined by a section $f\in H^0(\mathcal{O}_Y(d))$ and let $S=\underset{a\geqq 0}{H^0}(\mathcal{O}_Y(a))=\bigoplus_{a\geqq 0} S^a$ denote the homogeneous coordinate ring of Y and $\mathfrak{g}$ the Lie algebra of G.

Definition 1.0. Let Y be a compact Hermitian symmetric space which is not isomorphic to $\boldsymbol{P}^n$. The *Jacobian ideal* J_f of the smooth hypersurface X is a homogeneous ideal of S generated by $\mathfrak{g}\cdot f=\{v\cdot f\in S^d \mid v\in \mathfrak{g}\}$ and $f\in S^d$. The *Jacobian ring* $R=R_f$ is the quotient $R_f=S/J_f$.

Let Y, X, S, $f\in S^d$ and $R=R_f$ be as above. A positive integer $\lambda=\lambda(Y)$

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is defined by $K_Y=\mathcal{O}_Y(-\lambda)$ where K_Y denote the canonical bundle of Y. Put $N=\dim. Y$ and $\rho=\rho(Y, f)=(N+1)d-2\cdot\lambda$.

Definition 1.1. Let Y, $f\in S^d$ and $R=R_f$ be as above. We say that a pair of positive integers (d, a) satisfies the *condition* (DP) if

(i) $R^\rho\cong \boldsymbol{C}$, (ii) the natural pairing $R^a\otimes R^{\rho-a}\to R^\rho\cong\boldsymbol{C}$ is perfect.

Under these notation and definitions, we have the following theorem.

Theorem 1.2 (Duality theorem). (1) *Let Y be one of the following;* I. $Gr(m+n, n)$, $2\leqq n\leqq m$, *and* $(n, m)\neq(2, 2)$, II. $SO(2n)/U(n)$, $n\geqq 5$, III. $Sp(n)/U(n)$, $n\geqq 3$, IV. $Q^n(n\geqq 4)$, V. $E_6/Spin(10)\times T^1$, VI. $E_7/E_6\times T^1$. *Let (d, a) be a pair of positive integers such that $a\leqq 2d-\lambda$, $d\geqq\lambda$. Then the pair (d, a) satisfies the condition (DP) except for the cases where $Y=Q^n$, $a=d-2$ and $Y=Q^4$, $a=2d-4$.*

(2) *Let Y be Q^n $(n\geqq 4)$. Then we have the following exact sequences*

$$0\longrightarrow R^{d-2}\longrightarrow(R^{\rho-(d-2)})^*\longrightarrow\boldsymbol{C}\longrightarrow 0 \tag{1.1}$$

$$0\longrightarrow R^{2d-4}\longrightarrow(R^{\rho-(2d-4)})^*\longrightarrow\boldsymbol{C}\longrightarrow 0. \tag{1.2}$$

Definition 1.3. Let Y, $f\in S^d$ and $R=R_f$ be as in Definition 1.1 and let a, b be two positive integers such that $a<b$. We say that the triplet *(d, a, b) satisfies the condition (SL)* if the Koszul complex

$$R^{b-a}\longrightarrow(R^a)^*\otimes R^b\longrightarrow\Lambda^2(R^a)^*\otimes R^{a+b} \tag{1.3}$$

is exact at the middle term.

Theorem 1.4. *Let Y be one of the Hermitian symmetric spaces in Theorem 1.2. Let a, b be positive integers such that $a<b$. Then the triplet (d, a, b) satisfies the condition (SL) provided that $a+b\leqq 2d-\lambda$ and $d\geqq\lambda$, except for the case where $Y=Q^n$ and $b-a=d-2$.*

Remark 1.5. In the case where $Y=Q^n$, $d\geqq n$, $a+b\leqq 2d-n$ and $b-a=d-2$, the symmetrizer lemma does not hold. Put $T=\{P\in \mathrm{Hom}(R^a, R^b)\mid u\cdot P(v)=v\cdot P(u)$, for all $u, v\in R^a\}$. We have an exact sequence

$$0\longrightarrow R^{b-a}\longrightarrow T\longrightarrow\boldsymbol{C}\longrightarrow 0.$$

Remark 1.6. The positive integer $\lambda=\lambda(Y)$ is given as follows; I. $n+m$, II. $2n-2$, III. $n+1$, IV. n, V. 12, VI. 18.

2. Sketch of a proof of Theorem 1.2. Put $L=\mathcal{O}_Y(d)$ and let Σ_L denote the sheaf of germs of differential operators of L of order $\leqq 1$. By the symbol map $\Sigma_L\to\Theta_Y$, we have an exact sequence

$$0\longrightarrow\mathcal{O}_Y\longrightarrow\Sigma_L\longrightarrow\Theta_Y\longrightarrow 0 \tag{2.1}$$

whose extension class is given by $-2\pi\sqrt{-1}\cdot c_1(L)\in H^1(Y, \Omega^1_Y)$. As in [4], we define the section $\widetilde{df}\in H^0(\Sigma_L^*\otimes\mathcal{O}_Y(d))$ by the formula $\widetilde{df}\cdot D:=D\cdot f$ for $D\in H^0(\Sigma_L)$. Put

$$J_f^{a\prime\prime}:=\mathrm{Im.}\,(H^0(\Sigma_L(a-d))\xrightarrow{\widetilde{df}}H^0(\mathcal{O}_Y(a)))\subset S^a.$$

It is easy to see that for all $a\geqq d$, $J_f^{a\prime\prime}$ coincides with J_f^a. Using the section $\widetilde{df}\in H^0(\Sigma_L^*(d))$ we can construct Green's Koszul complex

$$0\longrightarrow\bigwedge^{N+1}\Sigma_L(-(N+1)d)\xrightarrow{\widetilde{df}}\bigwedge^N\Sigma_L(-Nd)\xrightarrow{\widetilde{df}}\cdots\bigwedge^p\Sigma_L(-pd)\cdots \tag{2.2}$$
$$\xrightarrow{\widetilde{df}}\Sigma_L(-d)\xrightarrow{\widetilde{df}}\mathcal{O}_Y\longrightarrow 0.$$

By this Koszul complex and the duality Theorem 2.15 in [4], we have the following theorem.

Theorem 2.1. *Let Y be a compact Hermitian space and $R=R_f$ the Jacobian ring of a smooth hypersurface $X=\{f=0\}$ of degree d in Y.*

(1) *If $H^p(\wedge^p\Sigma_L(-pd))=0$ for $1\leqq p\leqq N-1$ and $H^{p-1}(\wedge^p\Sigma_L(-pd))=0$ for $2\leqq p\leqq N$, then we have $R^\rho\cong \boldsymbol{C}$.*

(2) *Assume that $R^\rho\cong\boldsymbol{C}$. Let a be a non-negative integer satisfying the condition (DS): $H^p(\wedge^p\Sigma_L(a-pd))=0$, $1\leqq p\leqq N-1$, (resp. (DI): $H^{p-1}(\wedge^p\Sigma_L(a-pd))=0$, $2\leqq p\leqq N$). Then the natural pairing $R^a\otimes R^{\rho-a}\to R^\rho\cong\boldsymbol{C}$, gives the surjection $R^a\to(R^{\rho-a})^*\to 0$ (resp. the injection $R^a\to(R^{\rho-a})^*$).*

To prove the duality Theorem 1.2, we have only to check the conditions (DS) and (DI) in Theorem 2.1. Using the generalized Borel-Weil theorem (cf. [7], Theorem 6.4 and Corollary 8.2), we can prove the following proposition.

Proposition 2.2. *Let Y be one of the Hermitian symmetric spaces in Theorem 1.2 and assume that $d\geqq\lambda(Y)$.*

(i) *For all integers $a\leqq 2d-\lambda$, we have*

$$H^p(\wedge^{p-1}\Theta_Y(a-pd))=H^p(\wedge^p\Theta_Y(a-pd))=0,\qquad (1\leqq p\leqq N-1)$$

and

$$H^{p-1}(\wedge^{p-1}\Theta_Y(a-pd))=H^{p-1}(\wedge^p\Theta_Y(a-pd))=0,\qquad (2\leqq p\leqq N)$$

except for the case where $Y=Q^n$, $H^1(\Theta_Y(-2))\cong\boldsymbol{C}$ and $H^2(\wedge^2\Sigma_L(-4))\cong\boldsymbol{C}$.

The duality Theorem 1.2 is proved by using Theorem 1.2 and Proposition 2.2 and the following exact sequence; $0\to\wedge^{p-1}\Theta_Y\to\wedge^p\Sigma_L\to\wedge^p\Theta_Y\to 0$.

3. Sketch of a proof of Theorem 1.4. Put $V=H^0(\mathcal{O}_Y(1))$ and let $\tilde{S}=S(V)$ be the symmetric algebra of V. We have the surjection $\pi:\tilde{S}\to S$ and define $I=\ker\pi$, hence $S=\tilde{S}/I$. The ideal I is generated by quadrics, i.e., $I^2\otimes\tilde{S}^a\to I^{2+a}$ is surjection for all $a\geqq 0$ ([9]).

Lemma 3.1. *Let Y be as in Theorem 1.2. Let a, b be positive integers such that $a<b$. Then the Koszul complex*

$$\wedge^2R^a\otimes R^{\rho-(a+b)}\longrightarrow R^a\otimes R^{\rho-b}\longrightarrow R^{\rho-(b-a)}\tag{3.1}$$

is exact at the middle term provided that $\rho-b\geqq\max.(d,2)$.

The proof is similar to the one given in [3].

Dualizing the Koszul complex (1.3) and connecting the natural homomorphism $R^{\rho-i}\to(R^i)^*$, we have the following commutative diagram;

$$\begin{array}{llcccc}
(3.2) & \wedge^2R^a\otimes(R^{a+b})^* & \longrightarrow & R^a\otimes(R^b)^* & \longrightarrow & (R^{b-a})^* \\
& \uparrow & & H_1\uparrow & & H_2\uparrow \\
(3.3) & \wedge^2R^a\otimes R^{\rho-(a+b)} & \longrightarrow & R^a\otimes R^{\rho-b} & \longrightarrow & R^{\rho-(b-a)}.
\end{array}$$

In the case of Theorem 1.4, we have $\rho-b\geqq d>2$, therefore the bottom row is exact at the middle term by Lemma 3.1. By diagram-chasing, if H_1 is surjective and H_2 is injective, we get the exactness of (3.2) at the middle term. By applying Theorem 1.2 we have desired surjectivity and injectivity except for the case where $Y=Q^n$ and $b-a=d-2$. Note that by using the above diagram we can show that the symmetrizer lemma does not hold for the exceptional case (cf. Remark 1.5).

4. Generic Torelli Theorem. Using the symmetrizer lemma (Theorem 1.4), we can prove the following generic Torelli theorem.

Theorem 4.1. *Let Y be one of the Hermitian symmetric spaces in Theorem* 1.2. *Let M^d be the coarse moduli space of smooth hypersurfaces of degree d in Y and $P: M^d \rightarrow \Gamma \backslash D$ the period map (associated with the middle cohomology). Assuwe that $d > \lambda(Y)$ and $d \neq 2 \cdot \lambda(Y)$. Then the period map P is one to one on a Zariski open set of M^d except for the case where $Y = Q^n$ and $d = \lambda + 1$.*

Added in proof. After submitting this note, the author was informed that K. Konno obtained the result similar to Theorem 4.1.

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