86. On a Problem of R. Brauer on Zeta-Functions of Algebraic Number Fields

By Ken-ichi SATO
Faculty of Engineering, Nihon University

(Communicated by Shokichi IYANAGA, M. J. A., Dec. 12, 1985)

1. Introduction. Let $\zeta_K(s)$ denote the Dedekind zeta-function of an algebraic number field K. It has been shown by R. Brauer [3] that if Ω_1 and Ω_2 are two finite algebraic number fields which are both normal over their intersection k and their compositum is K, then

$$\zeta_K(s)\zeta_k(s)/\zeta_{\Omega_1}(s)\zeta_{\Omega_2}(s)$$

is an entire function. Let K_1 and K_2 be finite algebraic number fields over $k=K_1\cap K_2$. Suppose now that at least one of K_1, K_2 is non-normal over k, and $K=K_1K_2$. Does it happen that also in this case the function $\zeta_K(s)\zeta_k(s)/\zeta_{K_1}(s)\zeta_{K_2}(s)$ becomes an entire function? We call this question R. Brauer's problem, and show that it has positive answer in some cases.

2. Main theorems.

Theorem 1. *$\zeta_{Q(\sqrt[p]{n},\sqrt[p]{m})}(s)\zeta(s)/\zeta_{Q(\sqrt[p]{n})}(s)\zeta_{Q(\sqrt[p]{m})}(s)$ is an entire function of s, where p is an odd prime and n, m are p-free relatively prime rational integers.*

Proof. Let $\zeta=\exp(2\pi i/p)$. Then $\boldsymbol{Q}(\sqrt[p]{n},\zeta)/\boldsymbol{Q}$ is normal and $T=\mathrm{Gal}(\boldsymbol{Q}(\sqrt[p]{n},\zeta)/\boldsymbol{Q})$ is generated by the elements σ, τ as follows $\sigma^p=\tau^{p-1}=e$, $\tau\sigma\tau^{-1}=\sigma^g$, where g is a primitive root mod p and the elements σ and τ are characterized by $\sigma:\zeta\to\zeta$, $\sqrt[p]{n}\to\sqrt[p]{n}\zeta$, $\tau:\zeta\to\zeta^g$, $\sqrt[p]{n}\to\sqrt[p]{n}$. The group T has $p-1$ linear characters (i.e., irreducible characters of degree one) and precisely one simple non-linear character χ_p such that $\chi_p(e)=p-1$. Here $\chi_p(\rho)=-1$ for $\rho\in\langle\sigma\rangle-\{e\}$ and $\chi_p(\rho)=0$ for $\rho\notin\langle\sigma\rangle$. We consider the field $M=\boldsymbol{Q}(\sqrt[p]{n},\sqrt[p]{m},\zeta)$. Let τ^* be the element of $G=\mathrm{Gal}(M/\boldsymbol{Q})$ such that $\tau^*:\zeta\to\zeta^g$, $\sqrt[p]{n}\to\sqrt[p]{n}$, $\sqrt[p]{m}\to\sqrt[p]{m}$. Then $\Omega=\boldsymbol{Q}(\sqrt[p]{n},\sqrt[p]{m})$ is the intermediate field of M over $\boldsymbol{Q}$ fixed by the cyclic subgroup $H=\langle\tau^*\rangle\subset G$ so that $H=\mathrm{Gal}(M/\Omega)$. Next let δ be the element of $\mathrm{Gal}(M/\boldsymbol{Q})$ such that $\delta;\zeta\to\zeta$, $\sqrt[p]{n}\to\sqrt[p]{n}$, $\sqrt[p]{m}\to\sqrt[p]{m}\zeta$. Then $F=\boldsymbol{Q}(\sqrt[p]{n},\zeta)$ is the fixed field of $N=\langle\delta\rangle$ and we have $\mathrm{Gal}(\boldsymbol{Q}(\sqrt[p]{n},\zeta)/\boldsymbol{Q})\cong G/N$. Here we consider the map $G\xrightarrow{\varphi}G/N\xrightarrow{\chi_p}\boldsymbol{C}$. If we denote $\lambda_p(x)=\chi_p(\varphi(x))$, then λ_p is one of the irreducible characters of G. In particular, $\lambda_p(\tau^*)=\chi_p(\tau)=0$. Let 1_H be the principal character of H, and we denote by 1_H^G the induced character of G. $\lambda_p|_H$ denotes the restriction of λ_p to H. Frobenius reciprocity yields

$$(1_H^G,\lambda_p)_G=(1_H,\lambda_p|_H)_H=\frac{1}{p-1}\sum_{h\in H}\lambda_p|_H(h)$$
$$=\frac{1}{p-1}\Big\{\lambda_p|_H(e)+\sum_{e\neq h\in H}\lambda_p|_H(h)\Big\}=\frac{1}{p-1}\{(p-1)+0+0+\cdots+0\}=1.$$

On the other hand let σ^* be the element of G such that $\sigma^*; \zeta\to\zeta, \sqrt[p]{n}\to\sqrt[p]{n}\zeta, \sqrt[p]{m}\to\sqrt[p]{m}$, then $\boldsymbol{Q}(\sqrt[p]{m},\zeta)$ is the fixed field of $\langle\sigma^*\rangle$ and $\mathrm{Gal}(\boldsymbol{Q}(\sqrt[p]{m},\zeta)/\boldsymbol{Q})\cong G/\langle\sigma^*\rangle$. Here we consider the map $G\xrightarrow{\psi}G/\langle\sigma^*\rangle\xrightarrow{\chi_p'}C$, where χ_p' is the irreducible character of $\mathrm{Gal}(\boldsymbol{Q}(\sqrt[p]{m},\zeta)/\boldsymbol{Q})$ of degree $p-1$. Then the $\lambda_p'(x)=\chi_p'(\psi(x))$ is also the irreducible character of G and $(1_H^G,\lambda_p')_G=1$ holds. Therefore $1_H^G=1_G+\lambda_p+\lambda_p'+\sum n_i\chi_i$ holds, where the 1_G is the principal character of G and occurs with multiplicity 1, χ_i are non-principal irreducible characters ($\neq\lambda_p,\lambda_p'$) of G and the n_i are non-negative rational integers. At least one n_i is non-zero. By the theory of induced characters and the properties of the Artin L-functions we have,

$$\zeta_{\boldsymbol{Q}(\sqrt[p]{n})}(s)/\zeta(s)=L(s,\chi_p,F/\boldsymbol{Q})=L(s,\lambda_p,M/\boldsymbol{Q})$$

and

$$\zeta_{\boldsymbol{Q}(\sqrt[p]{m})}(s)/\zeta(s)=L(s,\chi_p',F_1/\boldsymbol{Q})=L(s,\lambda_p',M/\boldsymbol{Q}),\qquad\text{where } F_1=\boldsymbol{Q}(\sqrt[p]{m},\zeta).$$

It follows now

$$\begin{aligned}\zeta_\Omega(s)&=L(s,1_H,M/\Omega)=L(s,1_H^G,M/\boldsymbol{Q})\\&=L(s,1_G,M/\boldsymbol{Q})L(s,\lambda_p,M/\boldsymbol{Q})L(s,\lambda_p',M/\boldsymbol{Q})\prod L(s,\chi_i,M/\boldsymbol{Q})^{n_i}.\end{aligned}$$

Therefore $\zeta_\Omega(s)\zeta(s)/\zeta_{\boldsymbol{Q}(\sqrt[p]{n})}(s)\zeta_{\boldsymbol{Q}(\sqrt[p]{m})}(s)=\prod L(s,\chi_i,M/\boldsymbol{Q})^{n_i}$.

The direct product $\langle\delta\rangle\times\langle\sigma^*\rangle$ is a normal subgroup of G and $G/\langle\delta\rangle\times\langle\sigma^*\rangle\cong\mathrm{Gal}(\boldsymbol{Q}(\zeta)/\boldsymbol{Q})$ (the cyclic group of order $p-1$). On the other hand, $\langle\delta\rangle$ is a normal subgroup of G and $G/\langle\delta\rangle\cong\mathrm{Gal}(\boldsymbol{Q}(\sqrt[p]{n},\zeta)/\boldsymbol{Q})$. Thus G has a normal series : $G\supset\langle\delta\rangle\times\langle\sigma^*\rangle\supset\langle\delta\rangle\supset\{e\}$ all of whose factors are cyclic. Therefore G is a supersolvable group. Since every supersolvable group is a monomial group, the Artin L-function $L(s,\chi,K/k)$ is entire for every non-principal irreducible character χ of supersolvable group. (See K. Uchida [6], Theorem 1 and also R. W. van der Waall [7], p. 161). Theorem 1 follows.

Theorem 2. *$\zeta_{\boldsymbol{Q}(\sqrt[p]{n},\sqrt[q]{a})}(s)\zeta(s)/\zeta_{\boldsymbol{Q}(\sqrt[p]{n})}(s)\zeta_{\boldsymbol{Q}(\sqrt[q]{a})}(s)$ is an entire function where p,q are distinct odd primes, and n,a are p-free and q-free rational integers respectively.*

Proof. Let $\tilde{\tau}$ be the automorphism defined by the following action, $\tilde{\tau}:\zeta\to\zeta^g, \sqrt[p]{n}\to\sqrt[p]{n}, \xi\to\xi, \sqrt[q]{a}\to\sqrt[q]{a}$, where $\xi=\exp(2\pi i/q)$. As usual, $\tilde{\tau}|_{\boldsymbol{Q}(\sqrt[p]{n},\zeta)}$ denotes the restriction of $\tilde{\tau}$ to the field $\boldsymbol{Q}(\sqrt[p]{n},\zeta)$. Then $\tilde{\tau}|_{\boldsymbol{Q}(\sqrt[p]{n},\zeta)}=\tau$ holds. Similarly, let $\tilde{\rho}$ be the automorphism defined by the following action $\tilde{\rho}:\zeta\to\zeta, \sqrt[p]{n}\to\sqrt[p]{n}, \xi\to\xi^r, \sqrt[q]{a}\to\sqrt[q]{a}$, where r is a primitive root mod q. Then $\boldsymbol{Q}(\sqrt[p]{n},\sqrt[q]{a})$ is the invariant field of the direct product $H=\langle\tilde{\tau}\rangle\times\langle\tilde{\rho}\rangle$ considered as subgroup of $G^*=\mathrm{Gal}(\boldsymbol{Q}(\sqrt[p]{n},\sqrt[q]{a},\zeta,\xi)/\boldsymbol{Q})$. Let $L=\boldsymbol{Q}(\zeta,\xi,\sqrt[p]{n},\sqrt[q]{a})$, $M_1=\boldsymbol{Q}(\sqrt[p]{n},\zeta)$ and $M_2=\boldsymbol{Q}(\sqrt[q]{a},\xi)$.

$\Gamma=\mathrm{Gal}(L/M_1)$ is isomorphic to $T_2=\mathrm{Gal}(M_2/\boldsymbol{Q})$ and G^*/Γ is isomorphic to $T_1=\mathrm{Gal}(M_1/\boldsymbol{Q})$. Here we consider the map

$$G^*\xrightarrow{\phi}G^*/\Gamma=\mathrm{Gal}(\boldsymbol{Q}(\sqrt[p]{n},\zeta)/Q)\xrightarrow{\chi_p}\boldsymbol{C}.$$

$\psi_p(x)=\chi_p(\phi(x))$ is one of the irreducible characters of G^* (the so-called lifted character). The elements of $H=\langle\tilde{\tau}\rangle\times\langle\tilde{\rho}\rangle$ is $\tilde{\tau}^i\tilde{\rho}^j$, $0\leqq i\leqq p-2$, $0\leqq j\leqq q-2$.

$\psi_p(e)=p-1$. $\psi_p(\tilde{\tau}^i\tilde{\rho}^j)=\chi_p(\phi(\tilde{\tau}^i\tilde{\rho}^j))=\chi_p(\tau^i)=0$ for $i\neq 0$. $\psi_p(\tilde{\rho}^j)=\chi_p(\phi(\tilde{\rho}^j))$ $=\chi_p(e)=p-1$ for $i=0$, $1\leqq j\leqq q-2$.

$$(1_H^{G^*}, \psi_p)_{G^*}=(1_H, \psi_p|_H)_H=\frac{1}{(p-1)(q-1)}\{(p-1)+\overbrace{(p-1)+\cdots+(p-1)}^{q-2}\}=1.$$

Similarly $(1_H^{G^*}, \psi_q)_{G^*}=1$, where ψ_q is the lifted character of χ_q of $\mathrm{Gal}(\boldsymbol{Q}(\sqrt[q]{a}, \xi)/\boldsymbol{Q})$.

$$\zeta_{\boldsymbol{Q}(\sqrt[p]{n})}(s)/\zeta(s)=L(s, \chi_p, M_1/\boldsymbol{Q})=L(s, \psi_p, L/\boldsymbol{Q})$$
$$\zeta_{\boldsymbol{Q}(\sqrt[q]{a})}(s)/\zeta(s)=L(s, \chi_q, M_2/\boldsymbol{Q})=L(s, \psi_q, L/\boldsymbol{Q}).$$

Here $1_H^{G^*}=1_{G^*}+\psi_p+\psi_q+\sum m_i\theta_i$, where the θ_i $(\neq\psi_p, \psi_q)$ are non-principal irreducible characters of G^* and the m_i are non-negative rational integers, at least one of which is non-zero. Since $G^*\cong T_1\times T_2$ (the direct product) and T_1, T_2 are supersolvable group, G^* is a supersolvable group. Therefore each $L(s, \theta_i, L/\boldsymbol{Q})$ is entire.

$$\zeta_{\boldsymbol{Q}(\sqrt[p]{n}, \sqrt[q]{a})}(s)\zeta(s)/\zeta_{\boldsymbol{Q}(\sqrt[p]{n})}(s)\zeta_{\boldsymbol{Q}(\sqrt[q]{a})}(s)=\prod L(s, \theta_i, L/Q)^{m_i}$$

is an entire function in the whole complex plane.

References

[1] E. Artin: Über eine neue Art von L-Reihen. Abh. Math. Sem. Univ. Hamburg, **3**, 89–108 (1923).

[2] ——: Zur Theorie der L-Reihen mit allgemeinen Gruppen-charakteren. ibid., **8**, 292–306 (1930).

[3] R. Brauer: A note on zeta-functions of algebraic number fields. Acta Arith., **24**, 325–327 (1973).

[4] J. C. Lagarias and A. M. Odlyzko: On computing Artin L-functions in the critical strip. Math. Comp., **33**, 1081–1095 (1979).

[5] J. P. Serre: Linear Representations of Finite Groups. Springer (1977).

[6] K. Uchida: On Artin L-functions. Tôhoku Math. J., **27**, 75–81 (1975).

[7] R. W. van der Waall: A remark on the zeta-function of an algebraic number field. J. Reine Angew. Math., **266**, 159–162 (1974).