79. *Dedekind Domains which are not obtainable as Finite Integral Extensions of PID*

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All known Dedekind domains are obtainable as the integral closure of a suitable PID R in a finite extension of the quotient field of R. But the converse is not the case, as we shall see in this note (cf. Zariski-Samuel, [2], Chap. V, § 8).

§ 1. An example. Let K be a quadratic extension of the field of rational numbers $\boldsymbol{Q}$ whose class number is greater than 1. Let p be a rational prime number which is the product of two distinct prime elements in K, say, π and π' : $p=\pi\pi'$. The density theorem of prime ideals assures the existence of such a prime number.

Let S be the set of the elements π^n $(n\geqq 1)$. The set S is a multiplicative set of the ring A of algebraic integers in K. Let A_S be the quotient ring of A with respect to the set S. Then the ring $A_S=R$ is a Dedekind domain, since A is a Dedekind domain. It is easily seen that $A_S\cap \boldsymbol{Q}=\boldsymbol{Z}$ and the integral closure of $\boldsymbol{Z}$ in K is the ring A, which is a proper subring of A_S.

Next we show that the ring A_S is not a PID. For this purpose it suffices to prove that the ideal class group of A_S is isomorphic with that of A, which is the ideal class group of the field K. Let I_0 be the semigroup of ideals of A prime to the ideal $A\pi$, and I_S the semigroup of ideals of A_S. Consider the mapping $\mathfrak{a}\to\mathfrak{a}A_S$. This is clearly a bijection of I_0 onto I_S. Suppose $\mathfrak{a}A_s=(\alpha/\pi^k)A_S$ for some $\alpha\in A$ and a positive integer k. Since $\mathfrak{a}\subseteq(\alpha/\pi^k)A_S$, we have $\pi^s\mathfrak{a}\subseteq\alpha A$ for some integer s. As A is a Dedekind domain, there exists an ideal $\mathfrak{b}$ of A such that

$$\pi^S\mathfrak{a}=\alpha\mathfrak{b}. \tag{1}$$

Since $\alpha/\pi^k\in\mathfrak{a}A_S$, we have $\pi^t\alpha A\subseteq\mathfrak{a}$ for some integer t. By the same reason as above, we have an ideal $\mathfrak{c}$ of A satisfying

$$\pi^t\alpha A=\mathfrak{a}\mathfrak{c}. \tag{2}$$

From (1) and (2) we obtain $\pi^m A=\mathfrak{b}\mathfrak{c}$ for some m. This implies that the ideal $\mathfrak{b}$ divides the principal ideal $\pi^m A$. Thus we see that $\mathfrak{b}$ is principal, and so is $\mathfrak{a}$. Since any ideal class of A has a representative which is prime to $A\pi$, we have proved that the ideal class group of K is isomorphic with the ideal class group of A_S. Thus we have the following:

There exists a Dedekind domain R which is not obtained as the integral closure of $R\cap F$ in the quotient field K of R for any proper subfield F of K.

§2. Another possible example and a question. Let l be any fixed prime number. Let K be the cyclotomic $\boldsymbol{Z}_l$-extension over the field $\boldsymbol{Q}$, A the ring of algebraic integers in K, and A' the ring of l-integers in K, namely, $A'=\{l^{-s}x \mid x \in A, s \in N\}$. The ring A' is the integral closure in K of the ring of rational l-integers. Moreover we find that A' is a Dedekind domain and that the ideal class group of A' is isomorphic to $\varinjlim C_n$, where C_n is the ideal class group of the field K_n whose degree is l^n over $\boldsymbol{Q}$, since every prime number other than l is finitely decomposed in K, and the rings $A' \cap K_n$ are Dedekind domains. The inclusion map $C_n \to C_{n+1}$ is injective for all n, since the prime number l does not divide the order of C_n. Therefore A' is not a PID, provided the class number of K_n is greater than 1 for some n. Thus we have shown the following:

The quotient field K of the Dedekind domain A' has no proper subfield over which K is of finite degree. The integral domain A' is not a PID, provided there exists a positive integer n such that the class number of K_n is greater than 1.

However, as far as known for small values of l and n, the class number of the field K_n is 1. So we ask the following:

Are there a prime number l and a positive integer n such that the field K_n which is a cyclic extension of $\boldsymbol{Q}$ with degree l^n and with conductor l^{n+1} has the class number greater than 1?

Remark 1. For $l=2$, it is conjectured by H. Cohn that the class number of K_n is 1 for every n.

Remark 2. For any fixed prime number l it is known that the class number of K_n is bounded for all n ([1]).

References

[1] Washington, L.: The non-p-part of the class number in a cyclotomic Z_p-extensions. Invent. math., **49**, 87–97 (1979).

[2] Zariski, O. and Samuel, P.: Commutative Algebra I. Van Nostrand (1958).