61. On the Numerically Fixed Parts of Line Bundles

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The purpose of this paper is to study the base loci of line bundles. Details will appear elsewhere.

By V we denote a non-singular projective variety defined over an algebraically closed field k. For a line bundle L on V, we have the base locus $\mathrm{Bs}\,|L|$ of the complete linear system and the stable base locus $\mathrm{SBs}\,(L)=\bigcap_{m=1}^{\infty}\mathrm{Bs}\,|mL|$ (Fujita [1]). In this paper, by $\kappa_{num}(L, V)\geqq 0$, we mean that there exist a birational morphism $f: W\to V$, a positive integer m and a nef line bundle S on W such that $H^0(W, mf^*L-S)\neq 0$.

§0. Pseudo-effectivity. Let K stand for a field $\boldsymbol{Q}$ or $\boldsymbol{R}$. A K-1-*cycle* on V is an element of $Z_1(V)\otimes_{\boldsymbol{Z}} K$, where $Z_1(V)$ is a free abelian group generated by irreducible curves on V. A K-1-cycle C is said to be *nef* if $(D, C)\geqq 0$ for any irreducible divisor D on V. A K-line bundle L is said to be *pseudo-effective* if $(L, C)\geqq 0$ for any K-1-cycle C on V.

Proposition 0. *For any $\boldsymbol{Q}$-line bundle L on V, the following conditions are equivalent to each other:*

(1) *L is pseudo-effective.*

(2) *For any ample line bundle A on V, and for any integer $n\geqq 1$, we have $\kappa(A+nL, V)\geqq 0$.*

§1. The numerical base locus of L. We shall introduce the set $\mathrm{NBs}\,(L)$, which may be a numerical analog of $\mathrm{SBs}\,(L)$.

Proposition 1. *Let L be a $\boldsymbol{Q}$-line bundle and let A an ample $\boldsymbol{Q}$-line bundle. Then*

(1) $\mathrm{SBs}\,(A+nL)\subset\mathrm{SBs}\,(A+(n+1)L)$.

(2) $\bigcup_{n=1}^{\infty}\mathrm{SBs}\,(A+nL)$ *does not depend on the choice of A, depending only on L.*

Proof. (1) We take a sufficiently large m. Then mA is very ample and

$$\begin{aligned}\mathrm{SBs}\,(A+nL)&=\mathrm{Bs}\,|m(n-1)(A+nL)|\supset\mathrm{Bs}\,|mA+m(n-1)(A+nL)|\\&=\mathrm{Bs}\,|nm(A+(n-1)L)|=\mathrm{SBs}\,(A+(n-1)L).\end{aligned}$$

(2) Given two ample $\boldsymbol{Q}$-line bundles A_1 and A_2, we choose $p\gg 0$ such that pA_2-A_1 is very ample. For any $n\geqq 1$ and a sufficiently large $m\geqq 1$, we have

$$\begin{aligned}\mathrm{SBs}\,(A_1+pnL)&=\mathrm{Bs}\,|m(A_1+pnL)|\supset\mathrm{Bs}\,|m(pA_2-A_1)+m(A_1+pnL)|\\&=\mathrm{Bs}\,|mp(A_2+nL)|=\mathrm{SBs}\,(A_2+nL).\end{aligned}$$

By this,

$$\bigcup_{n=1}^{\infty} \mathrm{SBs}\,(A_1+nL)\supset \mathrm{SBs}\,(A_2+nL).$$

Exchanging A_1 and A_2, we complete the proof.

Definition. Noting the last proposition, we set

$$\mathrm{NBs}\,(L)=\bigcup_{n=1}^{\infty} \mathrm{SBs}\,(A+nL)$$

for an ample line bundle A, which is called a *numerical base locus.*

Proposition 2. (1) $\mathrm{NBs}\,(L)$ *is determined only by the numerical equivalence class of* L.

(2) $\mathrm{NBs}\,(L)=\phi$ *if and only if* L *is nef.*

(3) $\mathrm{NBs}\,(L)=V$ *if and only if* L *is not pseudo-effective.*

(4) $\mathrm{NBs}\,(L)\subset \mathrm{SBs}\,(L)$.

Proof. (1) Let L_1 and L_2 be $\boldsymbol{Q}$-line bundles such that L_1 is numerically equivalent to L_2. For an ample line bundle A, $A_1=A+n(L_1-L_2)$ is an ample $\boldsymbol{Q}$-line bundle, n being an arbitrary integer. Thus

$$\mathrm{SBs}\,(A+nL_1)=\mathrm{SBs}\,(A_1+nL_2)\subset \mathrm{NBs}\,(L_2)$$

by the last proposition.

Proofs of (2), (3), and (4) are easy.

§2. The numerical fixed part of L. We shall introduce the notion of the numerical fixed part of a $\boldsymbol{Q}$-line bundle L.

For a line bundle L and for an integer $m\geqq 1$, we denote by $F(m, L)$ the fixed part of $|mL|$ and a general member of $|mL|-F(m, L)$ is indicated by $M(m, L)$. Then

$$|mL|=|M(mL)|+F(m, L).$$

For any integer m and $p\geqq 1$, we have the inequality $pF(m, L)\geqq F(mp, L)$. So in $\mathrm{Div}\,(V)\otimes_{\boldsymbol{Z}} \boldsymbol{R}$ we can consider the lower bound of the sequence $(F(m, L)/m)_{m\in \boldsymbol{N}}$. Actually, this lower bound is given by

$$\lim_{m\to\infty} F(m\,!, L)/m\,!,$$

which is denoted by $F(L)$.

Proposition 3. *Let L be a pseudo-effective line bundle and let A be an ample line bundle.*

(1) $F(A+nL)/n\leqq F(A+(n+1)L)/(n+1)$.

(2) Let $F(A+nL)=\sum_{\Gamma} a_{\Gamma}(n\,; A, L)\Gamma$ *be an irreducible decomposition, where* $a(n\,; A, L)\in \boldsymbol{R}$ *and* Γ *is a prime divisor. Then*

$$a_{\Gamma}(A, L)=\lim_{n\to\infty} a_{\Gamma}(n\,; A, L)/n<\infty$$

and $\sum_{\Gamma} a_{\Gamma}(A, L)<\infty$.

Proof. (1) Similar to the Proposition 1 (1).

(2) Since $(A+nL, H^{d-1})\geqq \sum_{\Gamma} a_{\Gamma}(n\,; A, L)(\Gamma, H^{d-1})$ where $d=\dim V$ and H is an ample line bundle, we have the required results. Q.E.D.

We consider a divisor with countably many components

$$\sum_{\Gamma} a_{\Gamma}(A, L)\Gamma \in \prod_{\Gamma} \boldsymbol{R}_{\geq 0}\Gamma.$$

Proposition 4. $\sum_{\Gamma} a_{\Gamma}(A, L)\Gamma$ *depends only on the numerical equivalence class of* L.

Proof is similar to Proposition 1 (2).

Definition. Using the above notation, we set $NF(L)=\sum_{\Gamma} a_{\Gamma}(A, L)\Gamma$ for an ample line bundle A, which is called a *numerical fixed part* of L.

Remark. 1) Symbolically, we may write

$$NF(L)=\lim_{n\to\infty} F(A+nL)/n.$$

2) $NF(L)$ can be defined for any $\boldsymbol{Q}$-line bundle L.

Proposition 5. *$NF(L)$ is numerically fixed by L in Fujita's sense* (cf. [2]), i.e. *for any birational morphism $f: W\to V$ from a non-singular projective variety W over k and any effective Q-divisor E on W such that f^*L-E is nef, we have $E-f^*NF(L)$ is an effective $\boldsymbol{R}$-divisor. In particular, if $\kappa_{num}(L)\geqq 0$, then $NF(L)\in \mathrm{Div}\,(v)\otimes_{\boldsymbol{Z}} \boldsymbol{R}$.*

Proof. Since $m!\,(A+nf^*L)=m!\,(A+n(f^*L-E))+m!\,nE$, we have $m!\,nE\geqq F(m!, A+nf^*L)$. Thus $E\geqq NF(f^*L)$. Since $NF(f^*L)\geqq f^*NF(L)$ is easily checked, we obtain the required result. Q.E.D.

Proposition 6. *If L is a pseudo-effective $\boldsymbol{Q}$-line bundle on a non-singular algebraic surface, then $NF(L)$ coincides with the negative part of the Zariski decomposition of L* (cf. [4]).

Proposition 7. *Let L be a $\boldsymbol{Q}$-line bundle with $\kappa_{num}(L)\geqq 0$ on a projective variety V. Then for any irreducible curve C, if $(L-NF(L), C)<0$, then $C\subset \mathrm{NBs}\,(L)$.*

Theorem 8. *Assume that the characteristic of k is 0. If L is a pseudo-effective $\boldsymbol{Q}$-line bundle with $\kappa_{num}(L)\geqq 0$ on a non-singular projective 3-fold V, then there exists a birational morphism $f: W\to V$ from a non-singular projective variety V, such that $f^*L-NF(f^*L)$ is a pseudo-effective $\boldsymbol{R}$-divisor and nef in codimension* 1, i.e. *$\{C$ an irreducible curve; $(f^*L-NF(f^*L), C)<0\}$ is a finite set.*

References

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