56. A Characterization of Almost Automorphic Functions

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Recently R. A. Johnson gave us a linear almost periodic differential equation with an almost automorphic solution which is not almost periodic [1]. In this paper we study almost automorphic functions and obtain a characterization of them by using Veech's result and Levitan's N-almost periodic functions.

We denote the set of real numbers by R. Let X be a metric space with the metric d_X. A continuous mapping $\pi: X\times R\to X$ is called a *flow on (a phase space) X* if π satisfies following two conditions:

(1) $\pi(x, 0)=x$ for $x\in X$.

(2) $\pi(\pi(x, t), s)=\pi(x, t+s)$ for $x\in X$ and $t, s\in R$.

The orbit through $x\in X$ of π is denoted by $C_\pi(x)$. $M\subset X$ is called an invariant set of π if $C_\pi(x)\subset M$ for every $x\in M$. The restriction of π to an invariant set M of π is denoted by $\pi|M$. A non-empty compact invariant set M of π is called a *minimal set of π* if $\overline{C_\pi(x)}=M$ for every $x\in M$, where $\overline{C_\pi(x)}$ is the closure of $C_\pi(x)$. If X is itself a minimal set, we say that π is a *minimal flow on X*. A flow π is said to be *equicontinuous* if for each $\varepsilon>0$ there exists a $\delta>0$ such that $d_X(\pi(x, t), \pi(y, t))<\varepsilon$ for $x, y\in X$ with $d_X(x, y)<\delta$ and for $t\in R$. A point $x\in X$ is called an *almost automorphic point* if for each sequence $\{t_n\}\subset R$ there exists a subsequence $\{t_{n_k}\}\subset\{t_n\}$ such that $\pi(x, t_{n_k})\to y\in X$ and $\pi(y, -t_{n_k})\to x$ as $k\to\infty$. We denote the set of almost automorphic points of π by $A(\pi)$. We can easily see that if $x\in A(\pi)$, then $\overline{C_\pi(x)}$ is a minimal set of π, and that $A(\pi)$ is an invariant set of π. A minimal flow π is said to be *almost automorphic* if $A(\pi)\neq\phi$. Let π be a minimal flow on X. $\lambda\in R$ is called an *eigenvalue of π* if there exists a continuous function $\chi: X\to K$ such that the relation $\chi(\pi(x, t))=\chi(x)\exp(2\pi i\lambda t)$ holds for $(x, t)\in X\times R$, where K is the unit circle in the complex plane. In this case χ is called an *eigenfunction of π belonging to λ*. We denote the set of eigenvalues of π by $\Lambda(\pi)$. It is well known that $\Lambda(\pi)$ is a countable subgroup of R for any minimal flow.

Proposition 1. *Let π be an equicontinuous minimal flow on X. Then, if a sequence $\{t_n\}\subset R$ satisfies that* $\lim_{n\to\infty}\exp(2\pi i\lambda t_n)=1$ *for every $\lambda\in\Lambda(\pi)$, then we have $\pi(x, t_n)\to x$ as $n\to\infty$ for $x\in X$.*

Proof. We denote the eigenfunction of π belonging to $\lambda\in\Lambda(\pi)$ by χ_λ. Since π is equicontinuous, it is well known that, if $\chi_\lambda(x)=\chi_\lambda(y)$ $(x, y\in X)$ for every $\lambda\in\Lambda(\pi)$, then we have $x=y$. Let $x\in X$. We assume that

$\lim_{n\to\infty} \exp(2\pi i\lambda t_n)=1$ for every $\lambda \in \Lambda(\pi)$. Then we have $\lim_{n\to\infty} \chi_\lambda(\pi(x, t_n)) = \lim_{n\to\infty} \exp(2\pi i\lambda t_n)\chi_\lambda(x)=\chi_\lambda(x)$. If $\pi(x, t_{n_k})\to y'$ as $k\to\infty$ for some subsequence $\{t_{n_k}\}$ of $\{t_n\}$, then $\chi_\lambda(y')=\chi_\lambda(x)$ for every $\lambda \in \Lambda(\pi)$. This means, by the above, that $y'=x$. This implies that $\pi(x, t_n)\to x$ as $n\to\infty$.

Let π and ρ be flows on X and Y, respectively. A continuous mapping h of X onto Y is called a homomorphism from π to ρ if we have $h(\pi(x, t)) =\rho(h(x), t)$ for $(x, t)\in X\times R$.

Proposition 2. *Let π be a non-trivial almost automorphic minimal flow on X. Then there exist a non-trivial equicontinuous minimal flow ρ on Y and a homomorphism h from π to ρ such that $A(\pi)=\{x\in X\,;\, h^{-1}(h(x))=\{x\}\}$.*

Proof. See [3].

Let $C(R, R)$ be the set of real valued continuous functions with compact-open topology. We consider a flow η on $C(R, R)$ defined by $\eta(f, t)=f_t$ for $(f, t)\in C(R, R)\times R$, where $f_t(s)=f(t+s)$ for $s\in R$. A function $f\in C(R, R)$ is called an almost automorphic function if it is an almost automorphic point of η; that is, for every sequence $\{t_n\}\subset R$ there exists a subsequence $\{t_{n_k}\}\subset\{t_n\}$ such that $f_{t_{n_k}}\to g$ and $g_{-t_{n_k}}\to f$ as $k\to\infty$ in $C(R, R)$. We denote the hull of f, $\overline{\{f_t\}}_{t\in R}$, by $\Omega(f)$, and the restriction of η to $\Omega(f)$ by η_f. Let $f\in C(R, R)$. For $\varepsilon>0$ and $N>0$, put

$$E_{\varepsilon N}=\{\tau\,;\,|f(t+\tau)-f(t)|<\varepsilon \text{ for } |t|\leqq N\}.$$

We say that f is an N-almost periodic function if it satisfies the following conditions: For $\varepsilon>0$ and $N>0$

(1) $E_{\varepsilon N}$ is a relatively dense subset of R.

(2) There exists $\eta(\varepsilon, N)>0$ such that $E_{\eta N}\pm E_{\eta N}\subset E_{\varepsilon N}$.

Proposition 3. *Let $f\in C(R, R)$. f is an N-almost periodic if and only if there exists a countable subgroup Σ_f of R such that, if a sequence $\{t_n\}\subset R$ satisfies $\lim_{n\to\infty}\exp(2\pi i\lambda t_n)=1$ for every $\lambda\in\Sigma_f$, then we have $\eta(f,t_n) =f_{t_n}\to f$ as $n\to\infty$.*

Proof. See [2], p. 58.

Proposition 4. *If $f\in C(R, R)$ is an almost automorphic function, then f is an N-almost periodic function.*

Proof. Since this is stated in [2], p. 63 without the proof. We propose a proof. By the assumption η_f is an almost automorphic minimal flow on $\Omega(f)$. Hence, by Proposition 2, there exists an equicontinuous minimal flow ρ on Y, and a homomorphism h from η_f to ρ such that $A(\eta_f)=\{g\,;\,h^{-1}(h(g)) =\{g\}\}\ni f$. In Proposition 3, put $\Sigma_f=\Lambda(\rho)$, and assume that a sequence $\{t_n\}$ satisfies $\lim_{n\to\infty}\exp(2\pi i\lambda t_n)=1$ for each $\lambda\in\Lambda(\rho)$. Then, by Proposition 1, $\rho(h(f), t_n)\to h(f)$ as $n\to\infty$. Since $\rho(h(f), t_n)=h(\eta_f(f, t_n))=h(f_{t_n})$, if $f_{t_{n_k}}\to g$ for some subsequence of $\{t_n\}$, then $h(f)=h(g)$ by continuity of h. Hence $g=f$, because $f\in A(\eta_f)$. This means that $f_{t_n}\to f$ as $n\to\infty$. Hence, by Proposition 3, f is an N-almost periodic function.

Theorem 1. *Let $f\in C(R, R)$ be bounded and uniformly continuous on R. If f is N-almost periodic, then it is an almost automorphic function.*

Proof. It is enough to show that, if $f_{\alpha_n}\to g$ and $g_{-\alpha_n}\to h$ for some sequence $\{\alpha_n\}\subset R$, then $f=h$. Let $\varepsilon>0$ and $N>0$ be arbitrary. Since f is uniformly continuous on R, there exists a $\delta>0$ such that $|f(s)-f(t)|<\varepsilon$ for $s,\ t\in R$ with $|s-t|<\delta$. Choose a $\eta(\varepsilon, N)>0$ so that $E_{\eta N}\pm E_{\eta N}\subset E_{\varepsilon N}$, and let $l>0$ be an inclusion length of $E_{\eta N}$. Then we can represent $\alpha_n=\tau_n+s_n$ for each n, where $\tau_n\in E_{\eta N}$ and $|s_n|\leqq l$. We can assume $s_n\to s_0$ as $n\to\infty$. Choose a natural number N_1 so that $|s_n-s_m|<\delta$ for $n,\ m\geqq N_1$. Since $g_{-\alpha_n}\to h$ in $C(R, R)$ as $n\to\infty$, there exists a natural number $n_1\geqq N_1$ such that

$$|g_{-\alpha_{n_1}}(t)-h(t)|=|g(t-\alpha_{n_1})-h(t)|<\varepsilon$$

for $|t|\leqq N$. Similarly, since $f_{\alpha_n}\to g$ in $C(R, R)$ as $n\to\infty$, there exists a natural number $n_2\geqq N_1$ such that

$$|f_{\alpha_{n_2}}(t-\alpha_{n_1})-g(t-\alpha_{n_1})|=|f(t-\alpha_{n_1}+\alpha_{n_2})-g(t-\alpha_{n_1})|<\varepsilon$$

for $|t|\leqq N$. Then we have

$$\begin{aligned}&|f(t-\alpha_{n_1}+\alpha_{n_2})-h(t)|\\ &\leqq|f(t-\alpha_{n_1}+\alpha_{n_2})-g(t-\alpha_{n_1})|+|g(t-\alpha_{n_1})-h(t)|<2\varepsilon\end{aligned}$$

for $|t|\leqq N$. On the other hand, we have

$$\begin{aligned}&|f(t-\alpha_{n_1}+\alpha_{n_2})-f(t)|\\ &=|f(t-\tau_{n_1}+\tau_{n_2}+(s_{n_2}-s_{n_1}))-f(t)|\\ &\leqq|f(t-\tau_{n_1}+\tau_{n_2}+(s_{n_2}-s_{n_1}))-f(t-\tau_{n_1}+\tau_{n_2})|\\ &\quad+|f(t-\tau_{n_1}+\tau_{n_2})-f(t)|<2\varepsilon\end{aligned}$$

for $|t|\leqq N$. Hence we obtain

$$\begin{aligned}&|h(t)-f(t)|\\ &\leqq|h(t)-f(t-\alpha_{n_1}+\alpha_{n_2})|+|f(t-\alpha_{n_1}+\alpha_{n_2})-f(t)|<4\varepsilon\end{aligned}$$

for $|t|\leqq N$. Since ε and N are arbitrary, we conclude that $h=f$.

Theorem 2. *$f\in C(R, R)$ is an almost automorphic function if and only if there exist a non-trivial equicontinuous minimal flow ρ on Y, a real valued function Φ on Y and $y\in Y$ satisfying the following conditions:*

(1) *Φ is continuous on $C_\rho(y)$ with respect to the relative topology on Y.*

(2) *$f(t)=\Phi(\rho(y, t))$ for $t\in R$.*

(3) *f is bounded and uniformly continuous on R.*

Proof. Necessity: Since f is an almost automorphic function, it is bounded and uniformly continuous, and η_f is an almost automorphic minimal flow on $\Omega(f)$. Hence, by Proposition 2, there exist an equicontinuous minimal flow ρ on Y and a homomorphism from η_f to ρ satisfying $A(\eta_f)=\{g\in\Omega(f)\,;\,h^{-1}(h(g))=\{g\}\}$. Define a continuous function H on $\Omega(f)$ by $H(g)=g(0)$ for $g\in\Omega(f)$. Then $H(f_t)=f(t)$ for $t\in R$. We can easily see that the restriction of h to $A(\eta_f)$ is a homeomorphism from $A(\eta_f)$ to $h(A(\eta_f))$ with respect to relative topologies. Since $h(A(\eta_f))$ contains $h(f)$ and it is an invariant set of ρ, we have $C_\rho(h(f))\subset h(A(\eta_f))$. Define a function Φ on Y by

$$\Phi(y)=\begin{cases}H(h^{-1}(y)) & y\in h(A(\eta_f))\\ 0 & y\notin h(A(\eta_f)).\end{cases}$$

Then Φ is continuous on $C_\rho(h(f))$ with the relative topology, and $f(t)=\Phi(\rho(h(f), t))$ for $t\in R$.

Sufficiency: It is enough to show that f is an N-almost periodic function. In Proposition 3, put $\Sigma_f = \Lambda(\rho)$. Let a sequence $\{t_n\} \subset R$ satisfy $\exp(2\pi i \lambda t_n) \to 1$ as $n \to \infty$ for each $\lambda \in \Lambda(\rho)$. Then, by Proposition 1, we have $\rho(y, t_n) \to y$ as $n \to \infty$. Since Φ is continuous on $C_\rho(y)$ with respect to the relative topology, we can easily see that $f_{t_n} \to f$ as $n \to \infty$ uniformly on every compact subset of R. This implies, by Proposition 3, that f is an N-almost periodic function (cf. [2], p. 58).

References

[1] R. A. Johnson: A linear almost periodic equation with an almost automorphic solution. Proc. Amer. Math. Soc., **82**, 199–205 (1981).

[2] B. M. Levitan and V. V. Zhikov: Almost Periodic Functions and Differential Equations. Cambridge University Press (1982) (English translation).

[3] W. A. Veech: Almost automorphic functions on groups. Amer. J. Math., **87**, 719–751 (1965).