49. *Universal Central Extensions of Chevalley Algebras over Laurent Polynomial Rings and GIM Lie Algebras*

By Jun MORITA*) and Yōji YOSHII**)

(Communicated by Kôsaku YOSIDA, M. J. A., June 11, 1985)

We will give an explicit description of the universal central extensions of Chevalley algebras over Laurent polynomial rings with n variables, which is a natural generalization of the result for $n=1$ established in Garland [1], and which is obtained in a different way from Kassel [4]. Using this, we will discuss about a certain class of GIM Lie algebras which are introduced by Slodowy [5] as a generalization of Kac-Moody Lie algebras.

1. Central extensions of Chevalley algebras. Let F be a field of characteristic zero. For a finite dimensional split simple Lie algebra $\mathfrak{g}$ over F and an F-algebra R, we will write $\mathfrak{g}(R)=R\otimes_F \mathfrak{g}$ and view $\mathfrak{g}(R)$ as a Lie algebra over F. Since $\mathfrak{g}(R)=[\mathfrak{g}(R), \mathfrak{g}(R)]$, there is a unique, up to isomorphism, universal central extension of $\mathfrak{g}(R)$. A central extension of $\mathfrak{g}(R)$:

$$(1)\qquad 0\longrightarrow V\longrightarrow \mathfrak{a}\longrightarrow \mathfrak{g}(R)\longrightarrow 0$$

can be reduced to a skew-symmetric F-bilinear mapping:

$$(2)\qquad \{\cdot,\cdot\}: R\times R\longrightarrow V$$

satisfying $\{u, vw\}+\{v, wu\}+\{w, uv\}=0$ for all $u, v, w \in R$ (cf. [1], [3]).

2. Laurent polynomial rings. We denote by $F[X_1^{\pm 1}, \cdots, X_n^{\pm 1}]$ the ring of Laurent polynomials in $X_1, \cdots, X_n$ with coefficients in F. Let $\mathfrak{c}$ be the F-vector space with a basis $\{z_v^{(1)}, \cdots, z_v^{(n-1)}, z_0^{(n)} \mid v \in Z^n\}$. We define an F-bilinear mapping $\{\cdot,\cdot\}_1$:

$$F[X_1^{\pm 1}, \cdots, X_n^{\pm 1}]\times F[X_1^{\pm 1}, \cdots, X_n^{\pm 1}]\longrightarrow \mathfrak{c}$$

by, for all $r_i, s_i \in Z$ $(i=1, \cdots, n)$,

$$(3)\qquad \{X_1^{r_1}\cdots X_n^{r_n}, X_1^{s_1}\cdots X_n^{s_n}\}_1 = \begin{cases} \sum_{i=1}^{k-1}\dfrac{r_i s_k - s_i r_k}{l_k} z_v^{(i)} + \sum_{j=k+1}^{n} r_j z_v^{(j-1)} & \\ \qquad\qquad \text{if } l_k\neq 0,\ l_{k+1}=\cdots=l_n=0 \text{ for some } k, \\ \sum_{i=1}^{n} r_i z_0^{(i)} \qquad \text{if } l_i=0 \text{ for all } i, \end{cases}$$

where $v=(l_1, \cdots, l_n)$ and $l_i=r_i+s_i$.

Theorem 1. *Let $\mathfrak{g}$ be a finite dimensional split simple Lie algebra over F. Then the mapping* (3) *determines a universal central extension of* $\mathfrak{g}(F[X_1^{\pm 1}, \cdots, X_n^{\pm 1}])$.

Notice that the dimension of $\mathfrak{c}$ is one $(n=1)$; infinite $(n\geq 2)$.

3. n-fold extended Cartan matrices. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and $\mathfrak{g}=\mathfrak{h}+\sum_{\alpha\in\Delta}\mathfrak{g}^\alpha$ the root space decomposition of $\mathfrak{g}$ with respect to

*) Institute of Mathematics, University of Tsukuba.
**) Komaba Junior High School attached to University of Tsukuba.

$\mathfrak{h}$, where Δ is the root system of $(\mathfrak{g}, \mathfrak{h})$ and $\mathfrak{g}^\alpha$ is the root subspace of $\mathfrak{g}$ corresponding to $\alpha \in \Delta$. We choose a fundamental system $\Pi = \{\alpha_1, \cdots, \alpha_l\}$ of Δ, where $l = \dim \mathfrak{h}$. Let H_α be the coroot of α. We fix a Chevalley basis $\{H_{\alpha_i}, E_\alpha \mid 1 \leq i \leq l, \alpha \in \Delta\}$ of $\mathfrak{g}$ (cf. [2]). Here $H_{\alpha_i} \in \mathfrak{h}$ and $E_\alpha \in \mathfrak{g}^\alpha$. The Cartan subalgebra may be denoted by $\mathfrak{g}^0$. Let $Q = \sum_{i=1}^{l+n} \boldsymbol{Z}\tilde{\beta}_i$ be the free $\boldsymbol{Z}$-module generated by $\tilde{\beta}_1, \cdots, \tilde{\beta}_{l+n}$. For each $\tilde{\beta} = (k_1, \cdots, k_{l+n}) \in Q$, we define the subspace:

$$\mathfrak{g}(F[X_1^{\pm 1}, \cdots, X_n^{\pm 1}])^{\tilde{\beta}} = (FX_1^{k_{l+1}} \cdots X_n^{k_{l+n}}) \otimes_F \mathfrak{g}^{k_1\alpha_1 + \cdots + k_l\alpha_l}$$

of $\mathfrak{g}(F[X_1^{\pm 1}, \cdots, X_n^{\pm 1}])$. Then $\mathfrak{g}(F[X_1^{\pm 1}, \cdots, X_n^{\pm 1}])$ is a Q-graded Lie algebra over F. Let $\beta = b_1\alpha_1 + \cdots + b_l\alpha_l$ be the negative highest root of Δ with respect to Π. Put $\tilde{\alpha}_i = \tilde{\beta}_i$ $(1 \leq i \leq l)$ and $\tilde{\alpha}_j = b_1\tilde{\beta}_1 + \cdots + b_l\tilde{\beta}_l + \tilde{\beta}_j$ $(l+1 \leq j \leq l+n)$. Then $\{\tilde{\alpha}_i \mid 1 \leq i \leq l+n\}$ is a new basis of Q, and we will use this basis. Let $\alpha_j = \beta$ $(l+1 \leq j \leq l+n)$. We denote by $A^{[n]}$ the n-fold extended Cartan matrix of $\mathfrak{g}$, which is defined by

$$A^{[n]} = \left(2\frac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}\right)_{1 \leq i, j \leq l+n}.$$

4. GIM Lie algebras. An $m \times m$ integral matrix $B = (b_{ij})$ is called a generalized intersection matrix (=GIM) if

(i) $b_{ii} = 2$,

(ii) $b_{ij} < 0 \Longleftrightarrow b_{ji} < 0$,

(iii) $b_{ij} > 0 \Longleftrightarrow b_{ji} > 0$.

For each GIM $B = (b_{ij})$, we denote by $L(B)$ the Lie algebra over F generated by $e_1, \cdots, e_m, h_1, \cdots, h_m, f_1, \cdots, f_m$ with the defining relations: $[h_i, h_j] = 0$, $[h_i, e_j] = b_{ij}e_j$, $[h_i, f_j] = -b_{ij}f_j$, $[e_i, f_i] = h_i$ for all i, j, $[e_i, f_j] = (\mathrm{ad}\, e_i)^{|b_{ij}|+1}e_j = (\mathrm{ad}\, f_i)^{|b_{ij}|+1}f_j = 0$ for distinct i, j with $b_{ij} \leq 0$, $[e_i, e_j] = [f_i, f_j] = (\mathrm{ad}\, e_i)^{b_{ij}+1}f_j = (\mathrm{ad}\, f_i)^{b_{ij}+1}e_j = 0$ for distinct i, j with $b_{ij} > 0$. Then $L(B)$ is a $\boldsymbol{Z}^m$-graded Lie algebra over F in usual sense (cf. [5]).

Next suppose $B = A^{[n]}$. Then there is an extension

$$\phi : L(A^{[n]}) \longrightarrow \mathfrak{g}(F[X_1^{\pm 1}, \cdots, X_n^{\pm 1}])$$

defined by $\phi(e_i) = E_{\alpha_i}$, $\phi(e_{l+j}) = X_j \otimes E_\beta$, $\phi(h_i) = H_{\alpha_i}$, $\phi(h_{l+j}) = H_\beta$, $\phi(f_i) = E_{-\alpha_i}$, $\phi(f_{l+j}) = X_j^{-1} \otimes E_{-\beta}$ for all $i = 1, \cdots, l$ and $j = 1, \cdots, n$. Then the kernel J_1 of ϕ is homogeneous since both $L(A^{[n]})$ and $\mathfrak{g}(F[X_1^{\pm 1}, \cdots, X_n^{\pm 1}])$ are Q-graded. Furthermore J_1 is a maximal homogeneous ideal of $L(A^{[n]})$. Let $J = [L(A^{[n]}), J_1]$.

Theorem 2. *Notation is as above. Then $L(A^{[n]})/J$ is a universal central extension of $\mathfrak{g}(F[X_1^{\pm 1}, \cdots, X_n^{\pm 1}])$.*

Let I_1 be the maximal homogeneous ideal of $L(A^{[n]})$ which intersects the subspace of degree zero trivially. Put $I = I_1 + J$ and $L'(A^{[n]}) = L(A^{[n]})/I$. Then ϕ induces a central extension $\phi' : L'(A^{[n]}) \to \mathfrak{g}(F[X_1^{\pm 1}, \cdots, X_n^{\pm 1}])$. Let $V = \oplus_{i=1}^n Fv_i$, and let

$$\{\cdot, \cdot\}' : F[X_1^{\pm 1}, \cdots, X_n^{\pm 1}] \times F[X_1^{\pm 1}, \cdots, X_n^{\pm 1}] \longrightarrow V$$

be the mapping defined by

$$\{X_1^{r_1} \cdots X_n^{r_n}, X_1^{s_1} \cdots X_n^{s_n}\}' = \delta_{r_1, -s_1} \cdots \delta_{r_n, -s_n}(r_1v_1 + \cdots + r_nv_n).$$

Theorem 3. *Notation is as above. Then the central extension ϕ' is* *corresponding to the mapping* $\{\cdot, \cdot\}'$.

The algebra $L'(A^{[n]})$ is an n-fold generalization of the standard affine (Kac-Moody) Lie algebra $L(A^{[1]})=L'(A^{[1]})$.

5. A remark. Let (Ω, d) be the module of relative differential forms of $F[X_1^{\pm 1}, \cdots, X_n^{\pm 1}]$ over F. Using the fact that $F[X_1^{\pm 1}, \cdots, X_n^{\pm 1}]$ is a Hopf algebra, whose structure is given by $\Delta(X_i)=X_i \otimes X_i$ and $\varepsilon(X_i)=1$ for all $i=1, \cdots, n$, the module Ω is identified with $F[X_1^{\pm 1}, \cdots, X_n^{\pm 1}] \otimes_F I/I^2$, where I is the kernel of ε (cf. [6]). Then the mapping d is identified with $(1 \otimes \pi)\Delta$, where π is the composition of the projection from $F[X_1^{\pm 1}, \cdots, X_n^{\pm 1}]=F \oplus I$ to I and the canonical homomorphism: $I \rightarrow I/I^2$. Set $\Omega'=\Omega/d(F[X_1^{\pm 1}, \cdots, X_n^{\pm 1}])$. We define the mapping τ of $F[X_1^{\pm 1}, \cdots, X_n^{\pm 1}] \times F[X_1^{\pm 1}, \cdots, X_n^{\pm 1}]$ to Ω' by

$$\tau(u, v)=\overline{u d(v)} \tag{4}$$

for all $u, v \in F[X_1^{\pm 1}, \cdots, X_n^{\pm 1}]$. Then τ satisfies the condition of (2). The theory of Kassel [4] says that τ gives a universal central extension of $\mathfrak{g}(F[X_1^{\pm 1}, \cdots, X_n^{\pm 1}])$. It is easily seen that $(\mathfrak{c}, \{\cdot, \cdot\}_1)$ is equivalent to (Ω', τ) (cf. [7]).

Our results seem to be answers to some questions in [9, Section 4.15].

References

[1] H. Garland: The arithmetic theory of loop groups. Publ. Math. IHES, **52**, 5–136 (1982).

[2] J. E. Humphreys: Introduction to Lie Algebras and Representation Theory. Springer-Verlag, New York (1972).

[3] N. Jacobson: Lie Algebras. Interscience, New York (1962).

[4] C. Kassel: Kähler differentials and coverings of complex simple Lie algebras extended over a commutative algebra. J. Pure Appl. Algebra, (2–3), **34**, 265–275 (1984).

[5] P. Slodowy: Another new class of Lie algebras (preprint).

[6] W. C. Waterhouse: Introduction to Affine Group Schemes. Springer-Verlag, New York (1979).

[7] Y. Yoshii: Master thesis. Master's Program of Education, University of Tsukuba (1985) (in Japanese).

[8] P. Slodowy: Beyond Kac-Moody algebras, and inside (to appear in the Proceedings of the "Windsor Lie Algebra Conference". Windsor, Ontario (1984)) (a revised version of [5]).

[9] ——: Singularitäten, Kac-Moody-Liealgebren, assoziierte Gruppen und Verallgemeinerungen. Habilitationsschrift, Universität Bonn (1984–85).