47. The Existence of Spectral Decompositions in L^p-Subspaces

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1. Introduction. In this note we outline the main results of a forthcoming paper [4]. Throughout we suppose that μ is an arbitrary measure, $1<p<\infty$, and Y is a subspace of $L^p(\mu)$. An invertible operator $V\in\mathcal{B}(Y)$ will be called *power-bounded* provided $\sup_{n\in\boldsymbol{Z}}\|V^n\|<\infty$, where $\boldsymbol{Z}$ denotes the additive group of integers. We show that $\{V^n\}_{n=-\infty}^{\infty}$ is automatically the Fourier-Stieltjes transform of a spectral family of projections concentrated on $[0,2\pi]$ (see [1, §2] for definitions and the Riemann-Stieltjes integration theory of spectral families). We deduce that every bounded, one-parameter group on Y is the Fourier-Stieltjes transform of a spectral family of projections $E(\cdot):\boldsymbol{R}\to\mathcal{B}(X)$. This result generalizes work in [2], [8], and can be used to obtain a complete analogue for $L^p(\mathcal{K})$ of Helson's correspondence [10, § 2.3] between cocycles and the normalized, simply invariant subspaces of $L^2(\mathcal{K})$, where $\mathcal{K}$ is a compact abelian group with archimedean ordered dual. In particular, in $L^p(\mathcal{K})$ every such invariant subspace is the range of a bounded projection.

2. Abstract results. An operator U on a Banach space X is called *trigonometrically well-bounded* [3] provided

$$U=\int_{[0,2\pi]}^{\oplus} e^{i\lambda}dE(\lambda)$$

for a spectral family of projections $E(\cdot):\boldsymbol{R}\to\mathcal{B}(X)$ such that the strong left-hand limits $E(0^-)$, $E((2\pi)^-)$ are 0, I, respectively. $E(\cdot)$ is necessarily unique, and will be called the *spectral decomposition* of U. Let $BV(\boldsymbol{T})$ be the Banach algebra of complex-valued functions having bounded variation on the unit circle. For $f\in BV(\boldsymbol{T})$ put

$$F_1(t)=\lim_{s\to t+} f(e^{is}),\qquad F_2(t)=\lim_{s\to t-} f(e^{is})$$

for $t\in\boldsymbol{R}$, and let $\hat{f}$ be the Fourier transform of f.

(2.1) **Theorem.** *Let $U\in\mathcal{B}(X)$ be trigonometrically well-bounded and power-bounded, and suppose $f\in BV(\boldsymbol{T})$. Then $\sum_{n=-N}^{N}\hat{f}(n)U^n$ converges in the strong operator topology, as $N\to+\infty$, to*

$$2^{-1}\int_{[0,2\pi]}^{\oplus}(F_1+F_2)dE,$$

where $E(\cdot)$ is the spectral decomposition of U.

Proof. For $t\in\boldsymbol{R}$, $x\in X$, let

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(2.2) $$\Phi(t)x=\int_{[0,2\pi]}^{\oplus} f(e^{it}e^{i\lambda})dE(\lambda)x,$$

and put

$$\hat{\Phi}(n)=(2\pi)^{-1}\int_0^{2\pi} e^{-int}\Phi(t)xdt$$

for $n\in \boldsymbol{Z}$. If we replace $\Phi(t)x$ in the second integral by the right of (2.2) and interchange the order of integration, we obtain $\hat{\Phi}(n)x=\hat{f}(n)U^n x$. This step requires further justification, however, since $E(\cdot)$ is not given by a measure. By [6, Lemma 17.2 and proof of 17.4] the approximating sums for the integral in (2.2) converge uniformly in t, and this fact legitimizes the foregoing argument. The vector-valued versions of Fejér's Theorem and a standard Tauberian theorem of Hardy [11, Theorems I.3.1, II.2.2] together with [6, Theorem 17.5] can now be applied to $\Phi(t)x$ at $t=0$ to give the conclusion of (2.1) readily.

Henceforth the convergence of a series $\sum_{n=-\infty}^{\infty} u_n$ will signify that of the "balanced" partial sums, $\sum_{n=-N}^{N} u_n$, and $\|\cdot\|_{\boldsymbol{T}}$ will denote the norm of $BV(\boldsymbol{T})$.

(2.3) **Corollary.** *Under the hypotheses of Theorem* (2.1):

(i) *there is a constant* C_U *such that*

(2.4) $$\|\textstyle\sum_{n=-N}^{N}\hat{f}(n)U^n\|\le C_U\|f\|_{\boldsymbol{T}}, \qquad \text{for } N\ge 0,\ f\in BV(\boldsymbol{T});$$

(ii) *for* $0\le\lambda<2\pi$, $x\in X$,

(2.5) $$E(\lambda)x=\textstyle\sum_{k=-\infty}^{\infty}\hat{g}_\lambda(k)U^k x+\lim_n (2n)^{-1}\sum_{k=0}^{n-1}e^{-ik\lambda}U^k x \\ +\lim_n (2n)^{-1}\sum_{k=0}^{n-1}U^k x,$$

where $g_\lambda\in BV(\boldsymbol{T})$ *is the characteristic function of* $\{e^{it}: 0\le t\le\lambda\}$.

Proof. Standard considerations with the Fourier series of $\Phi(t)x$ show that

$$\sum_{k=-n}^{n}\left(1-\frac{|k|}{n+1}\right)\hat{f}(k)U^k x=(2\pi)^{-1}\int_0^{2\pi}K_n(t)\Phi(t)xdt,$$

where $\{K_n\}$ is Fejér's kernel. Since $f\in BV(\boldsymbol{T})$, $|\hat{f}(k)|\le(2\pi|k|)^{-1}\operatorname{var}(f,\boldsymbol{T})$, for $k\ne 0$. The conclusion in (2.4) is immediate from these facts and application of [1, Proposition (2.3)] to (2.2). By Theorem (2.1) the series on the right of (2.5) is $2^{-1}\{E(\lambda^-)+E(\lambda)-E(0)\}x$. The functional calculus described in [1, Proposition (2.3)] can be used to show that the second term in (2.5) is $2^{-1}\{E(\lambda)-E(\lambda^-)\}x$. We omit the details.

3. Spectral decomposition of power-bounded operators on Y. Throughout this section V will denote a power-bounded operator on the subspace Y of $L^p(\mu)$, as set forth in §1. We put $c=\sup_{n\in\boldsymbol{Z}}\|V^n\|$.

(3.1) **Transference lemma.** *For any trigonometric polynomial*

$$Q(z)\equiv\textstyle\sum_{n=-N}^{N}a_n z^n (z\in\boldsymbol{T}),\ \|Q(V)\|\le c^2\|Q\|_{p,p},$$

where $\|Q\|_{p,p}$ *is the* $L^p(\boldsymbol{Z})$*-multiplier norm of* Q.

Proof. The demonstration is a special case, for the group $\boldsymbol{Z}$ and the representation $n\mapsto V^n$, of the proof in [5, Theorem 2.4].

(3.2) **Theorem.** *V is trigonometrically well-bounded, and*

$$\sup\{\|E(\lambda)\|:\lambda\in\boldsymbol{R}\}\le A_p c^2,$$

where $E(\cdot)$ is the spectral decomposition of V, and A_p is a constant depending only on p.

Proof. Application of Stečkin's Theorem [7, Theorem 6.4.4] to Theorem (3.1) shows that V has a continuous $AC(\boldsymbol{T})$-functional calculus, where $AC(\boldsymbol{T})$ is the subalgebra of $BV(\boldsymbol{T})$ consisting of all absolutely continuous functions. By [3, Theorem 2.3], V is trigonometrically well-bounded, and $\sup\{\|E(\lambda)\| : \lambda \in \boldsymbol{R}\} \leq 3c^2\alpha_p$, where α_p is the constant of Stečkin's Theorem.

(3.3) **Corollary.** *For $f \in BV(\boldsymbol{T})$, $\|\sum_{n=-\infty}^{\infty}\hat{f}(n)V^n\| \leq c^2\|f\|_{p,p}$.*

Proof. Let $\sigma_N(f, V)$ be the N^{th} Cesàro mean for $\sum_{n=-\infty}^{\infty}\hat{f}(n)V^n$, and put $Q_N = K_N * f$. Thus $\sigma_N(f, V) = Q_N(V)$, and so $\|\sigma_N(f, V)\| \leq c^2\|Q_N\|_{p,p} \leq c^2\|f\|_{p,p}$. Let $N \to +\infty$ and apply Theorem (2.1).

(3.4) **Corollary.** *V has a logarithm belonging to $\mathcal{B}(Y)$.*

(3.5) **Corollary.** *Every hermitian-equivalent operator T on Y is well-bounded.*

Proof. The hypothesis (see [6, p. 108]) is that e^{iT} is power-bounded. Theorem (3.2) and the proof in [6, Theorem 20.28] now give the conclusion.

Remarks. (i) Theorem (3.2) generalizes theorems in [9] and [12] concerning translation operators. (ii) If Y is replaced by an arbitrary reflexive space, the first assertion in Theorem (3.2), as well as Corollary (3.4), fails [4, (5.1), (5.4)].

(3.6) **Theorem.** *If $\{V_t\}$, $t \in \boldsymbol{R}$, is a strongly continuous, one-parameter group of operators on Y such that $\sup_{t\in\boldsymbol{R}}\|V_t\| < \infty$, then there is a unique spectral family $E(\cdot)$ of projections in Y such that*

$$V_t y = \lim_{a\to+\infty}\int_{-a}^{a} e^{it\lambda}dE(\lambda)y, \qquad \text{for } y \in Y,\ t \in \boldsymbol{R}.$$

Moreover, $\{V_t : t \in \boldsymbol{R}\}$ and $\{E(\lambda) : \lambda \in \boldsymbol{R}\}$ have the same commutants.

Proof. By Theorem (3.2), $\{V_t\}$ satisfies the hypotheses of [1, Theorem (4.20)].

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