37. CR-Microfunctions and the Henkin-Ramirez Reproducing Kernel

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§ 0. The method of integral representations is natural and very useful in the theory of functions of several complex variables. One of the most important integral representations was given by Henkin [4] and by Ramirez [9]. They succeeded in constructing a holomorphic reproducing kernel for strictly pseudo-convex domains by using the so-called Cauchy-Fantappiè formula.

On the other hand, the curvilinear wave expansion (Radon transformation) due to Sato-Kawai-Kashiwara, which is a natural generalization of the plane wave decomposition of the δ-function, is a fundamental tool in the theory of linear partial differential equations. For, by using this expression, one can reproduce holomorphic functions which define a given hyper (or micro) function (Kataoka [6]). Bony [2] noticed that this curvilinear wave expansion was constructed from the Cauchy-Fantappiè kernel by taking its boundary values to the real domain as local cohomology.

In this paper, we study the boundary values of the Henkin-Ramirez kernel from the microlocal point of view and verify in particular that the Henkin-Ramirez kernel is a holomorphic reproducing kernel of CR-microfunctions (=microfunction solutions of the tangential Cauchy-Riemann equations). We also show at the same time that the way of construction of the Henkin-Ramirez kernel and that of Sato-Kawai-Kashiwara's curvilinear wave expansion of the δ-function are essentially of the same type from the microlocal point of view.

This simple observation makes it possible in a unified manner to construct holomorphic reproducing kernels for CR-microfunctions on a certain class of CR-submanifolds. This generalization will be dealt in the subsequent paper.

§ 1. Let D be a strictly pseudo-convex domain in C^n with real analytic boundary ∂D; $D=\{z\,|\,\rho(z)<0\}$, where ρ is a real analytic defining function such that $d\rho(z)$ does not vanish on ∂D. Since we are interested in the (micro-)local properties, we only consider the Levi polynomial L_ρ associated with ρ, which is an analytic family of local holomorphic support functions defined by

$$L_\rho(z, z')=\sum_{j=1}^{n}\frac{\partial\rho}{\partial z'_j}(z')(z'_j-z_j)-\frac{1}{2}\sum_{j,k=1}^{n}\frac{\partial^2\rho}{\partial z'_j\partial z'_k}(z')(z'_j-z_j)(z'_k-z_k).$$

Lemma 1.

(i) *For each* $z' \in \partial D$, $L_\rho(z, z')$ *is holomorphic in* z.

(ii) *For each* $z' \in \partial D$, *there is a neighborhood* U *of* z' *such that* $L_\rho(z, z') \neq 0$ *for* $z \in (\bar{D} - \{z'\}) \cap U$.

Remark 2 (Schapira [12]). $\{z \mid L_\rho(z, z') = 0\}$ is an analytic family of positive complex hypersurfaces with respect to $T^*_{\partial D}(C^n)$.

Set $P = (P_1, P_2, \cdots, P_n)$, where

$$P_j = \frac{\partial \rho}{\partial z'_j}(z') - \frac{1}{2} \sum_{k=1}^{n} \frac{\partial^2 \rho}{\partial z'_j \partial z'_k}(z')(z'_k - z_k), \qquad j = 1, 2, \cdots, n.$$

By the definition, $\langle z' - z, P \rangle = L_\rho(z, z')$ and P satisfies Leray's condition for D (see [8]). Moreover, we have the following

Lemma 3 (Henkin [4], Ramirez [9]).

(i) $\mathrm{Re}\,(L_\rho(z, z')) \geqq \rho(z') - \rho(z) + \mathrm{const.}\ |z - z'|^2$ *in some neighborhood.*

(ii) $\partial_z L_\rho(z, z')|_{z=z'} = \partial_{z'} \rho(z')$.

Recall that a real analytic function φ defined on a real analytic manifold N is said to be of positive type at $p \in N$, if $\varphi(p) \neq 0$ or if $\mathrm{Re}\,(d\varphi(p))$ is a non-zero covector and $\mathrm{Re}\,(\varphi(q)) = 0$ implies $\mathrm{Im}\,(\varphi(q)) \geqq 0$ for every q sufficiently near p (see [11], Chap. 1, Def. 3.1.4).

Lemma 3 implies

Lemma 4 (cf. Schapira [12]).

(i) $iL_\rho(z, z')|_{\partial D \times \partial D}$ *is a function of positive type.*

(ii) $\mathrm{Re}\,(d_{x,y}(iL_\rho(z, z')|_{\partial D \times \partial D}))|_{z=z'} = \mathrm{Im}\,(\partial_{z'} \rho(z'))$.

(iii) $\mathrm{Im}\,(d(iL_\rho(z, z')|_{\partial D \times \partial D}))|_{z=z'} = 0$.

§ 2. Let us consider the boundary behavior of the following Cauchy-Fantappiè type kernel (= local Henkin-Ramirez kernel):

$$K(z, z') = \frac{(n-1)!}{(2\pi i)^n} \sum_{k=1}^{n} (-1)^{k-1} \frac{P_k dP_{[k]} \wedge dz'}{\langle z' - z, P \rangle^n} \qquad \text{for } (z, z') \in D \times \partial D,$$

where $dP_{[k]} = dP_1 \wedge dP_2 \wedge \cdots \wedge dP_{k-1} \wedge dP_{k+1} \wedge \cdots \wedge dP_n$.

For this purpose, we first determine the singular spectrum (analytic wave front set) of $b_{\partial D} K(z, z')$, where $b_{\partial D} K(z, z')$ denotes the boundary value of $K(z, z')$ to $\partial D \times \partial D$.

Proposition 5. *The singular spectrum of* $b_{\partial D} K(z, z')$ *coincides with the anti-diagonal set on the boundary*; *that is,*

$$SS(b_{\partial D} K(z, z')) = \{(z', z'; k d\rho(z'), -k d\rho(z')) \mid z' \in \partial D, k > 0\}.$$

Hence, by the standard procedure, we can define a microlocal operator associated with this kernel.

Let $T^*_{\partial D}(C^n)^+$ (resp. $T^*_{\partial D}(C^n)^-$) be the exterior (resp. interior) conormal bundle. We regard $T^*_{\partial D}(C^n)^\pm$ as subsets of $\sqrt{-1}T^*(\partial D)$. Our main result is the following.

Theorem 6 (cf. Sato-Kawai-Kashiwara [11], Chap. 3, Example 1.2.5).

(i) *For any microfunction* f *with* $\mathrm{supp}\,(f) \subseteqq \sqrt{-1}T^*(\partial D) - T^*_{\partial D}(C^n)^-$, *the boundary value*

$$b_{\partial D} \int_{\partial D} K(z, z') f(z') = \int_{\partial D} b_{\partial D} K(z, z') f(z')$$

is a well-defined CR-*microfunction.*

(ii) *For any* CR-*microfunction f on the exterior conormal bundle* $T^*_{\partial D}(C^n)^+ =\{(z', kd\rho(z'))\,|\, z' \in \partial D, k>0\}$, *we get*

$$f(z)=b_{\partial D}\int_{\partial D} K(z, z')f(z')=\int_{\partial D} b_{\partial D}K(z, z')f(z').$$

As the next example shows, the Henkin-Ramirez kernel is a natural generalization of the Sato kernel which was used to determine the structure of microfunction solutions of H. Lewy's equation.

Example 7 (Sato [10], Sato-Kawai-Kashiwara [11], Chap. 1, Example 3.2.4). Set $D=\{(z_1, z_2) \in C^2\,|\, \rho=x_2+x_1^2+y_1^2<0\}$. By a direct calculation, we get

$$L_\rho(z, z')=\bar{z}_1'(z_1'-z_1)+\frac{1}{2}(z_2'-z_2).$$

We can easily verify that the microlocal operator associated with $L_\rho(z, z')$ is equal to the one given by M. Sato.

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