14. A Remark on the Global Markov Property for the d-Dimensional Ising Model

By Yasunari HIGUCHI
Department of Mathematics, Faculty of Science, Kobe University

(Communicated by Kôsaku YOSIDA, M. J. A., Feb. 13, 1984)

1. Introduction. Let Z^d be the d-dimensional cubic lattice and $\Omega\equiv\{-1,+1\}^{Z^d}$ be the configuration space, equipped with the product of discrete topology. $\mathcal{F}$ stands for the Borel σ-field of Ω. The sub σ-fields $\{\mathcal{F}_V; V\subset Z^d\}$ are defined by

$$\mathcal{F}_V\equiv\sigma\{\omega(x); x\in V\}.$$

A probability measure μ on $(\Omega,\mathcal{F})$ is said to have *local Markov property* (LMP), if for every finite $V\subset Z^d$,

$$\mu(\cdot\,|\mathcal{F}_{V^c})(\omega)=\mu(\cdot\,|\mathcal{F}_{\partial V})(\omega) \qquad \text{on } \mathcal{F}_V\ \mu\text{-a.s. } \omega, \tag{1}$$

where $\partial V\equiv\{x\in V^c; |x-y|\equiv\max\{|x^i-y^i|; 1\leqq i\leqq d\}=1$ for some $y\in V\}$. If (1) holds for any $V\subset Z^d$, then μ is said to have *global Markov property* (GMP). It is known that (LMP) does not necessarily imply (GMP) (see for example, [4], [6], [7]). Therefore the question is when (LMP) implies (GMP). In this note, we discuss this question for the d-dimensional Ising model. The Hamiltonian of this model is given for each finite $V\subset Z^d$, with magnetic field h, and the boundary condition $\omega\in\Omega$, by

$$E_V(\eta|\omega)=\sum_{x,y\in V}J_{x,y}\eta(x)\eta(y) + \sum_{x\in V}\sum_{y\in\partial V}J_{x,y}\eta(x)\omega(y)+h\sum_{x\in V}\eta(x), \tag{2}$$

where $J_{x,y}=J_{0,|x-y|}=0$ unless $|x-y|=1$. For $\beta>0$, the corresponding finite Gibbs state for (2) is given by

$$P_{\beta,V}(\{\eta(x),\ x\in V\}|\omega)=(\text{normalization})\cdot\exp\{-\beta E_V(\eta|\omega)\}. \tag{3}$$

and

$$P_{\beta,V}(\{\eta(x)=\omega(x),\ x\in V^c\}|\omega)=1. \tag{4}$$

A Gibbs state for the Ising model (2) is a probability measure μ on $(\Omega,\mathcal{F})$ satisfying

$$\mu(\cdot\,|\mathcal{F}_{V^c})(\omega)=P_{\beta,V}(\cdot\,|\omega)\ \mu\text{-a.s. } \omega, \qquad \text{for every finite } V\subset Z^d. \tag{5}$$

By definition, any Gibbs state for Ising model (2) has (LMP), but not every Gibbs state for (2) has (GMP) (a counterexample is given in [4]). If $J=\{J_{x,y}\}$ satisfies Dobrushin's uniqueness condition, then the unique Gibbs state has (GMP) ([2], [3]).

In this note, we assume that the Ising model (2) has attractive interaction; $J_{x,y}\leqq 0$ for every pair $x,y\in Z^d$. In this case, it is known that there exists a critical β_c, $0<\beta_c\leqq\infty$ (the last equality holds iff $d=1$), such that Gibbs state is unique for $\beta<\beta_c$, and non-unique for

$\beta>\beta_c$. Moreover, there exist extremal Gibbs states μ_+ and μ_- satisfying

(6) $\qquad E_{\mu_+}(f)\geqq E_\mu(f)\geqq E_{\mu_-}(f) \qquad$ if f is increasing,

for any Gibbs state μ, where we define the order in Ω by the componentwise inequality; $\omega\geqq\eta$ iff $\omega(x)\geqq\eta(x)$ for any $x\in Z^d$.

It is proved that both μ_+ and μ_- have (GMP), and henceforth the unique Gibbs state has (GMP) whenever $\beta<\beta_c$ ([3], [4]). It seems to be natural to expect that every extremal Gibbs state has (GMP) (see [4]), but unfortunately there is no answer to this question. Here, we give a new class of Gibbs states for Ising model (2) with attractive interaction, which has (GMP).

Theorem 1. *For every* $\alpha\in[0,1]$, *let* $\mu_\alpha\equiv\alpha\mu_++(1-\alpha)\mu_-$. *Then,* μ_α *has* (GMP).

In the simplest case, i.e. $d=2$, and $J_{x,y}=-\delta_{|x-y|,1}$, it is known that every Gibbs state for (2) equals μ_α for some $\alpha\in[0,1]$ ([1], [5]), which implies

Theorem 2. *For* $d=2$, $J_{x,y}=-\delta_{|x-y|,1}$, *every Gibbs state for* (2) *has* (GMP).

2. Proof of Theorem 1. We start with the following:

Lemma. *Let* μ_1,μ_2 *be distinct translation-invariant, mixing probability measures on* $(\Omega,\mathcal{F})$, *i.e.*

(7) *for any finite* $V_1, V_2\subset Z^d$,

$$\lim_{|r|\to\infty}\sup_{A\in\mathcal{F}_{V_1}}\sup_{B\in\mathcal{F}_{V_2+r}}|\mu_i(A\cap B)-\mu_i(A)\mu_i(B)|=0 \qquad (i=1,2),$$

(8) $\qquad \mu_i(A)=\mu_i(\tau_rA) \qquad$ *for any* $A\in\mathcal{F}$, $r\in Z^d$ $(i=1,2)$,

where $\tau_r:\Omega\to\Omega$ *is defined by* $(\tau_r\omega)(x)=\omega(x+r)$, $x\in Z^d$.

Let R *be a finite subset of* Z^d, *such that there exists an event* $A^*\in\mathcal{F}_R$ *with* $\mu_1(A^*)\neq\mu_2(A^*)$. *If* $S\subset Z^d$ *satisfies that*

(9) $$\sharp\{r\in Z^d\,;\,R+r\subset S\}=\infty,$$

then $\mathcal{F}_S$ *separates* μ_1 *and* μ_2, *i.e. there exists an event* $D\in\mathcal{F}_S$ *such that* $\mu_1(D)=1=\mu_2(\Omega\setminus D)$.

Proof. We enumerate the set $\{r\in Z^d\,;\,R+r\subset S\}$ by $\{r_1,r_2,\cdots\}$ and define a sequence of random variables $X_1,X_2,\cdots$ by

$$X_k(\omega)=I_{[\tau_{-r_k}A^*]}(\omega).$$

Obviously, X_k is $\mathcal{F}_{R+r_k}$-measurable. Therefore substituting R both for V_1 and V_2 in (7), for any $\varepsilon>0$ we can find $r_0\geqq0$ sufficiently large so that

$$|E_{\mu_i}(X_kX_j)-\mu_i(A^*)^2|<\varepsilon \qquad \text{if } |r_k-r_j|\geqq r_0\ (i=1,2).$$

This implies the $L^2(\mu_i)$-convergence of $n^{-1}\sum_{1\leqq k\leqq n}X_k$, $i=1,2$. In fact, we have

$$\begin{aligned}
&E_{\mu_i}|n^{-1}\textstyle\sum_{1\leqq k\leqq n}X_k-\mu_i(A^*)|^2\\
&\quad=n^{-2}\textstyle\sum_{1\leqq k,j\leqq n}\{E_{\mu_i}(X_kX_j-\mu_i(A^*)^2)\}\\
&\quad=n^{-2}\textstyle\sum_{1\leqq k\leqq n}\{\sum_{1\leqq j\leqq n,|r_k-r_j|<r_0}+\sum_{1\leqq j\leqq n,|r_k-r_j|\geqq r_0}\}\\
&\quad\leqq n^{-1}(2r_0+1)^d+\varepsilon \quad (i=1,2).
\end{aligned}$$

Thus, taking a subsequence $n_1, n_2, \cdots$, we obtain that

$$n_p^{-1} \sum_{1 \leqq k \leqq n_p} X_k(\omega) \to \mu_i(A^*) \qquad \text{as } p \to \infty \ \mu_i\text{-a.s. } (i=1,2).$$

Putting

$$D \equiv \{\omega \in \Omega\,;\, \lim_{p \to \infty} \{n_p^{-1} \sum_{1 \leqq k \leqq n_p} X_k(\omega)\} = \mu_1(A^*)\},$$

we obtain the desired result. Q.E.D.

Proof of Theorem 1. The statement of Theorem 1 is trivial if Gibbs state is unique. Therefore we can assume that $\mu_+ \neq \mu_-$, i.e. $d \geqq 2$, and $\beta > \beta_c$.

1) If ∂V is a finite set, then either V or V^c is a finite set. Since μ_α has (LMP), (1) holds if V is a finite set. So assume that V^c is a finite set. Let W be any finite subset of V, and $U \equiv V^c \backslash \partial V = (V \cup \partial V)^c$. Fix $\eta \in \Omega$ arbitrarily and let $A \in \mathcal{F}_U$, $B \in \mathcal{F}_{\partial V}$ be atoms such that

$$A = \{\omega \in \Omega\,;\, \omega(x) = \eta(x),\ x \in U\}, \qquad B = \{\omega \in \Omega\,;\, \omega(x) = \eta(x),\ x \in \partial V\}.$$

Then by definitions (3) and (5), we have for any $C \in \mathcal{F}_W$,

$$\mu_\alpha(A \cap B \cap C) = \int_{B \cap C} P_{\beta,U}(A \mid \omega) \mu_\alpha(d\omega) = P_{\beta,U}(A \mid \eta) \mu_\alpha(B \cap C),$$

which implies that

$$\mu_\alpha(C \mid A \cap B) = \mu_\alpha(A \cap B \cap C) / \mu_\alpha(A \cap B) = \mu_\alpha(C \mid B).$$

Since the last term of the above equality equals $\mu_\alpha(C \mid \mathcal{F}_{\partial V})(\eta)$, and since $W \subset V$, $C \in \mathcal{F}_W$, and $\eta \in \Omega$ are arbitrary, we obtain (1).

2) If ∂V is an infinite set, then we apply the lemma. Since μ_+ and μ_- are known to be translation-invariant and mixing, and since $\mu_+(\omega(0)=1) \neq \mu_-(\omega(0)=1)$, i.e. any infinite set S satisfies (9), we can find an event $D \in \mathcal{F}_{\partial V}$ such that $\mu_+(D) = 1 = \mu_-(\Omega \backslash D)$. Let f be an $\mathcal{F}_V$-measurable bounded function, and g be an $\mathcal{F}_{V^c}$-measurable bounded function. By definition, we have

$$E_{\mu_\alpha}(fg) = \alpha E_{\mu_+}(fg) + (1-\alpha) E_{\mu_-}(fg) = \alpha E_{\mu_+}(f^+ g) + (1-\alpha) E_{\mu_-}(f^- g),$$

where $f^+(\omega) = E_{\mu_+}(f \mid \mathcal{F}_{\partial V})(\omega)$, $f^-(\omega) = E_{\mu_-}(f \mid \mathcal{F}_{\partial V})(\omega)$, since μ_+ and μ_- have (GMP). Noting that

$$E_{\mu_+}(f^+ g) = E_{\mu_+}\{(f^+ I_D) g\} = \alpha^{-1} E_{\mu_\alpha}\{(f^+ I_D) g\}$$

and

$$E_{\mu_-}(f^- g) = (1-\alpha)^{-1} E_{\mu_\alpha}\{(f^- I_{\Omega \backslash D}) g\},$$

we obtain

$$E_{\mu_\alpha}(fg) = E_{\mu_\alpha}\{(f^+ I_D + f^- I_{\Omega \backslash D}) \cdot g\},$$

which proves (1). Q.E.D.

References

[1] Aizenman, M.: Translation invariance and instability of phase coexistence in the two dimensional Ising system. Comm. Math. Phys., **73**, 83–94 (1980).

[2] Alveberio, S., and Høegh-Krohn, R.: Uniqueness and global Markov property for Euclidean fields and lattice systems. Quantum Fields-Algebras, Processes (ed. Streit, L.). Springer, pp. 303–330 (1980).

[3] Föllmer, H.: On the global Markov property. ibid., pp. 293–302 (1980).

[4] Goldstein, S.: Remarks on the global Markov property. Comm. Math. Phys., **74**, 223–234 (1980).

[5] Higuchi, Y.: On the absence of the non-translation invariant Gibbs states for the two-dimensional Ising model. Colloquia Math. Societatis Janos Bolyai, vol. 27, Random Fields, Esztergom (Hungary) 1979 (ed. Fritz, J. *et al.*), North-Holland, pp. 517–534 (1981).

[6] Künsch, H.: Gaussian Markov random fields. J. Fac. Sci. Univ. of Tokyo, Sec. 1A, **26**, 53–73 (1979).

[7] Weizsäcker, H. V.: Exchanging the order of taking suprema and countable intersections of σ-algebras. Annals de l'I.H.P., Sec. B, **19**, 91–102 (1983).