92. *Remark on Nilpotency of Derivations*

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Let R be a ring and ∂ be a derivation of R. ∂ is called *nilpotent* if $\partial^n R=0$ for some positive integer n. The nilpotency of ∂ is the smallest such n. Let R be a semiprime ring. It was proven in [1] and [2] that the nilpotency of ∂ is power of 2 if R is of characteristic 2, and it is an odd number if R is 2-torsion free. The results are best possible when the characteristic is 2 or 0. When the characteristic is an odd prime, the results can be sharpened. In this note we give a precise result in case of positive characteristic and complete the study of nilpotency of derivations in semiprime rings.

Theorem. *Let R be a semiprime ring of prime characteristic p and ∂ be a nilpotent derivation of R. Then the nilpotency n of ∂ is of the following form:*

(*) $$n=\alpha_L P^L+\alpha_{L+1}P^{L+1}+\cdots+\alpha_M P^M,$$

where $0\leqq L\leqq M$, α_i are nonnegative integers less than p, α_L is odd and $\alpha_{L+1},\cdots,\alpha_M$ are even.

Proof. When $p=2$, the theorem asserts that n is a power of 2, this agrees with the results in [2]. Let $p\geqq 3$ and write $n=\beta_0+\beta_1 p+\cdots+\beta_M p^M$ with $0\leqq\beta_i<p$, $0\leqq M$ and $\beta_M\neq 0$. Let L be the greatest among the i's such that β_i is odd. Let $D=\partial^{p^L}$, then D is also a derivation of R and $D^l=0$, where $l=1+\beta_L+\beta_{L+1}p+\cdots+\beta_M p^{M-L}$. Since l is even, $\partial^{p^L(l-1)}=D^{l-1}=0$ by the main theorem in [1]. It follows that

$$n\leqq p^L(l-1)=\beta_L p^L+\beta_{L+1}p^{L+1}+\cdots+\beta_M p^M.$$

This implies $n=\beta_L p^L+\beta_{L+1}p^{L+1}+\cdots+\beta_M p^M$ and β_L is odd and $\beta_{L+1},\cdots,\beta_M$ are even, as desired.

The theorem is best possible in the following sense: For any number n of the form (*), there exist a prime ring R of characteristic p and a nilpotent derivation ∂ of R with nilpotency n. In fact, let $m=[n/2]+1$, where $[n/2]$ is the greatest integer not exceeding $n/2$. Let R be the $m\times m$ matrix ring over a field of characteristic p and let $A=E_{12}+E_{23}+\cdots+E_{m-1,m}$, where E_{ij} is the matrix with 1 at (i, j)-position and 0 elsewhere. Let $\partial=ad_A$ be the inner derivation induced by A. We claim that ∂ has nilpotency n. If p is odd, then $n=2m-1$. Since $A^m=0$, we have $\partial^n=0$. On the other hand ∂^{n-1} is not zero,

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because $\partial^{n-1}X=(-1)^{m-1}\binom{n-1}{m-1}A^{m-1}XA^{m-1}$ for any $X\in R$ and $\binom{n-1}{m-1}\not\equiv 0 \pmod p$ by [1, Lemma 3]. If $p=2$, then $n=2m-2$. We have $\partial^n X=(-1)^{m-1}\binom{n}{m-1}A^{m-1}XA^{m-1}=0$ for any $X\in R$, because $\binom{n}{m-1}$ is even. Since $A^{m-1}\neq 0$, there is $Y\in R$ such that $A^{m-1}YA^{m-1}\neq 0$. Hence $\partial^{n-1}(AY)=(-1)^{m-1}\binom{n-1}{m-1}A^{m-1}YA^{m-1}\neq 0$, because $\binom{n-1}{m-1}$ is odd. In either case the nilpotency of ∂ is n.

More generally, the argument above combined with Theorem shows

Corollary 1. *Let R be a semiprime ring of prime characteristic p. Let a be a nilpotent element in R of nilpotency m. Then the nilpotency of the inner derivation induced by a is the greatest integer of the form (*) not exceeding $2m-1$.*

If the characteristic q of a semiprime ring R is positive but not necessarily prime, then q is a product of distinct primes $p_1, p_2, \cdots, p_r$ and $R=R_1\oplus R_2\oplus\cdots\oplus R_r$, where $R_i=\{x\in R \mid p_i x=0\}$. R_i is a semiprime ring of prime characteristic p_i and $\partial R_i\subseteqq R_i$ for any derivation ∂ of R. Therefore, the nilpotency of ∂ is equal to the maximum of the nilpotencies of the restrictions of ∂ to R_i $(i=1, 2, \cdots, r)$. Hence by Theorem we have

Corollary 2. *Let R be a semiprime ring of positive characteristic q. Then the nilpotency of a nilpotent derivation of R is of the form (*) for some prime divisor p of q.*

References

[1] L. O. Chung and J. Luh: Nilpotency of derivations. Canad. Math. Bull., **26**, 341–346 (1983).

[2] ——: Nilpotency of derivations. II. Proc. Amer. Math. Soc., **91**, 357–358 (1984).