73. *Table of the Fourier Coefficients of Eisenstein Series of Degree 3*

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§1. **Introduction.** Let $\mathcal{H}_n$ be the Siegel upper-half space of degree n, and Z the variable on $\mathcal{H}_n$, then Eisenstein series $\psi_k(Z)$ of degree n and weight k is defined by

$$\psi_k(Z)=\sum |CZ+D|^{-k},$$

where C and D run over complete representatives of the equivalent classes of $n\times n$ square, coprime and symmetric pairs of integral matrices. $\psi_k(Z)$ can be expanded to the Fourier series:

$$\psi_k(Z)=\sum_{T\geqq 0} a_k(T)\, e^{2\pi i\sigma(TZ)}.$$

In [2], we have given explicit formulas for the Fourier coefficients $a_k(T)$ of $\psi_k(Z)$ in the case when $n=3$ and T are positive definite semi-integral primitive ternary matrices. By means of these formulas, we have calculated the numerical values of $a_k(T)$ for $k\leqq 24$ and for "smaller" T's. In the following, we give a table of such values which will be useful for the arithmetical investigations. (For technical reasons, this gives only a part of our result whose scope will be specified below. We are willing to communicate our result to anyone interested.)

§2. **Organization and usage of the table.** Concerning the positive definite semi-integral primitive ternary symmetric matrices we can utilize the table by Brandt-Intrau [1]. In their table the matrix

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ a_{23} & a_{13} & a_{12} \end{pmatrix},$$

is used to indicate the ternary quadratic form

$$f=a_1x_1^2+a_2x_2^2+a_3x_3^2+a_{12}x_1x_2+a_{13}x_1x_3+a_{23}x_2x_3.$$

From this we obtain a matrix $T_f=(b_{ij})$, where $b_{ii}=a_i$ $1\leqq i\leqq 3$ and $b_{ij}=b_{ji}=a_{ij}/2$ $1\leqq i<j\leqq 3$. If f is a positive definite ternary quadratic form with integer coefficients, then T_f is a positive definite semi-integral ternary matrix. In [1] d means $(-1)/2\times\det(2T_f)$. However we prefer to use $d=d_f$ to denote $\det(2T_f)/2$ which is positive. Since by [2] we know that the Fourier coefficient $a_k(T)$ of Eisenstein series of degree 3 is a genus invariant, we employ as the repesentative of each genus the first form given in [1]. Our table consists of Tables I and II.

In Table I, we present a table of ternary quadratic forms according to the order of the quantity d. When there is unique genus for a given d, we simply assign the index $T(d)$ to the representative form. When there are many genera for a given d, we assign the numbered indices $T_i(d)$ to the representative forms according to the order in which they appear in [1]. To each form we indicate the 2-adic type as 2-type, which is explained in [2], This Table is limited to $d \leqq 14$, though our calculation has been made for $d \leqq 50$.

Table I

d	reduced form	index, 2-type	d	reduced form	index, 2-type	d	reduced form	index, 2-type
2	$\begin{smallmatrix}1&1&1\\0&1&1\end{smallmatrix}$	$=T(2)$, $D(2)$	3	$\begin{smallmatrix}1&1&1\\0&0&1\end{smallmatrix}$	$=T(3)$, $D(2)$	4	$\begin{smallmatrix}1&1&1\\0&0&0\end{smallmatrix}$	$=T(4)$, $D(3)$
5	$\begin{smallmatrix}1&1&2\\0&1&1\end{smallmatrix}$	$=T(5)$, $D(1)$	6	$\begin{smallmatrix}1&1&2\\1&1&0\end{smallmatrix}$	$=T_1(6)$, $D(1)$	6	$\begin{smallmatrix}1&1&2\\0&0&1\end{smallmatrix}$	$=T_2(6)$, $D(2)$
7	$\begin{smallmatrix}1&1&2\\0&1&0\end{smallmatrix}$	$=T(7)$, $D(2)$	8	$\begin{smallmatrix}1&1&3\\0&1&1\end{smallmatrix}$	$=T_1(8)$, $D(2)$	8	$\begin{smallmatrix}1&1&2\\0&0&0\end{smallmatrix}$	$=T_2(8)$, $D(3)$
9	$\begin{smallmatrix}1&1&3\\0&0&1\end{smallmatrix}$	$=T(9)$, $D(1)$	10	$\begin{smallmatrix}1&1&3\\1&1&0\end{smallmatrix}$	$=T_1(10)$, $D(2)$	10	$\begin{smallmatrix}1&2&2\\2&0&1\end{smallmatrix}$	$=T_2(10)$, $D(1)$
11	$\begin{smallmatrix}1&1&3\\0&1&0\end{smallmatrix}$	$=T(11)$, $D(1)$	12	$\begin{smallmatrix}1&1&4\\0&0&1\end{smallmatrix}$	$=T_1(12)$, $D(2)$	12	$\begin{smallmatrix}1&2&2\\0&1&1\end{smallmatrix}$	$=T_2(12)$, $D(1)$
12	$\begin{smallmatrix}1&1&3\\0&0&0\end{smallmatrix}$	$=T_3(12)$, $D(3)$	12	$\begin{smallmatrix}1&2&2\\2&0&0\end{smallmatrix}$	$=T_4(12)$, $D(3)$	13	$\begin{smallmatrix}1&2&2\\1&0&1\end{smallmatrix}$	$=T(13)$, $D(1)$
14	$\begin{smallmatrix}1&1&4\\1&1&0\end{smallmatrix}$	$=T_1(14)$, $D(1)$	14	$\begin{smallmatrix}1&1&5\\0&1&1\end{smallmatrix}$	$=T_2(14)$, D(1)			

In Table II, we give the values of the Fourier coefficient $a_k(T_i(d))$ of Eisenstein series of degree 3 and weight k within the range $4 \leqq k \leqq 24$ and $k \equiv 0 \pmod 2$ with $k \neq 8$. (For other values of k, we have $a_k(T)=0$.) There we use $d-i$ to indicate $T_i(d)$. The values are modified as follows. Since for $k=4$ and 6 $a_k(T)$ are integer valued, we factor out the common divisor h_k of $a_k(T)$: $h_4=120960$, $h_6=(-1)\times 133056$. For example, the true value of $a_4(T(2))$ is $120960\times 3=362880$ and the true value of $a_6(T(3))$ is $(-1)\times 133056\times 80=-10644480$. We omit $a_8(T)$ from the table, because by virtue of Igusa-Kneser's Theorem generalizing Witt's identity we have $\psi_8(Z)=\psi_4^2(Z)$, which implies $a_8(T)=\sum_{T_1+T_2=T} a_4(T_1)a_4(T_2)$. For $10 \leqq k \leqq 24$, the values of $a_k(T)$ are not integral, so we factor out the common fractional part h_k of $a_k(T)$. h_k are given by

$h_{10}=(-1)\times 7584192/43867$,

$h_{12}=36167040/53678953$,

$h_{14}=(-1)\times 576/657931$,

$$h_{16}=2804820480/3617\cdot 1723168255201,$$
$$h_{18}=(-1)\times 689472/43867\cdot 151628697551,$$
$$h_{20}=316800/174611\cdot 154210205991661,$$
$$h_{22}=(-1)\times 83740608/77683\cdot 1520097643918070802691,$$
$$h_{24}=147813120/103\cdot 2294797\cdot 25932657025822267968607.$$

Table II

T \ k	4	10	14	22
2	3	255	4095	1048575
3	8	6560	531440	3486784400
4	15	65535	16777215	109 9511627775
5	24	390624	244140624	9536 7431640624
6-1	40	1685920	2177309680	365616 1925798800
6-2	30	1673310	2176254990	365615 4954327150
7	48	5764800	1 3841287200	7979226 6297612000
8-1	51	16711935	6 8702703615	115292040 5096267775
8-2	63	16777215	6 8719476735	115292150 4606846975
9	80	43046720	28 2429536480	1215766545 9056928800
10-1	78	99609630	99 9755863470	9999990463 2569407590
10-2	120	100390368	100 0244136528	1 0000009536 7430592048
11	120	214358880	313 8428376720	6 7274999493 2560009200
12-1	104	428243360	891 3907424240	38 3375633519 3011264400
12-2	168	431602080	891 8260980720	38 3376364750 9889293200
12-3	136	429922720	891 6084202480	38 3375999135 1450278800
12-4	150	430040670	891 6116694030	38 3375999354 3499965550
13	168	815730720	2329 8085122480	190 0496377488 0799438800
14-1	240	1481553600	5670 7753658400	836 6833521755 1098124000
14-2	150	1470024510	5668 0071092190	836 6817563301 8504997150

T \ k	6	18	24
2	15	65535	4194303
3	80	43046720	3 1381059608
4	255	4294967295	1759 2186044415
5	624	15 2587890624	238418 5791015624
6-1	1360	282 1152888640	13162173 5219132440
6-2	1230	282 1066926270	13162167 2465401830
7	2400	3323 2930569600	390982104 8582988048
8-1	3855	28147 0681808895	7378695870 2656356351
8-2	4095	28147 4976710655	7378697629 4838206463
9	6560	185302 0188851840	9 8477090218 3611232880
10-1	9390	999984 7412174910	99 9999761581 4213178678
10-2	10608	1000015 2587825088	100 0000238418 5786821320
11	14640	4594972 9863572160	814 0274938683 9761113320
12-1	19280	18488143 3533273920	5520 6130727317 1774535704
12-2	21840	18488707 5752957760	5520 6157051657 9450681368
12-3	20560	18488425 4643115840	5520 6143889487 5612608536
12-4	20910	18488426 3146956990	5520 6143892999 7222578150
13	28560	66541660 9183179840	32118 3887795485 5105157368
14-1	40800	217798657 0739875200	163989 8197317686 9684666640
14-2	36030	217792010 4878867070	163989 7415353477 2527079150

Table II (continued)

T \ k	12	16	20
2	1023	16383	262143
3	59048	4782968	387420488
4	1048575	268435455	6 8719476735
5	9765624	6103515624	381 4697265624
6–1	60524200	7 8368930680	10156 0343826760
6–2	60408150	7 8359397510	10155 9569510070
7	282475248	67 8223072848	162841 3597910448
8–1	1072694271	439 7778092031	1801432 9790267391
8–2	1073741823	439 8046511103	1801439 8509481983
9	3486784400	2287 6792454960	15009463 5296999120
10–1	9990235398	9999 3896500758	99999618 5302996518
10–2	1 0009764600	10000 6103499240	100000381 4697003480
11	2 5937424600	37974 9833583240	555991731 3492231480
12–1	6 1855909544	128383 9836748664	2662323165 2596781384
12–2	6 1976839848	128399 6565044088	2662343477 2509593928
12–3	6 1916374696	128391 8200896376	2662333321 2553187656
12–4	6 1918353750	128391 8728201350	2662333334 9217300150
13	13 7858491848	393737 6385699288	1 1245540695 1957393128
14–1	28 9537129200	1111268 5048614480	4 2688048262 4234390960
14–2	28 8972180750	1111132 8602501550	4 2687722579 7039094350

References

[1] H. Brandt and O. Intrau: Tabellen reduzierter positiver ternärer quadratischer Formen. Akademie-Verlag (1958).

[2] M. Ozeki and T. Washio: Explicit formulas for the Fourier coefficients of Eisenstein series of degree 3 (to appear in Crelle's J.).

[3] C. L. Siegel: Einführung in die Theorie der Modulfunktionen n-ten Grades. Math. Ann., **116**, 617–657 (1939).