69. On the Asymptotic Behavior of a Nonlinear Contraction Semigroup and the Resolvent Iteration

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1. Introduction. Throughout this note X denotes a real Banach space, A is an m-dissipative operator in X and $\{T(t): t \geqq 0\}$ is the contraction semigroup on $\overline{D(A)}$ (the closure of the domain of A) generated by A. For $r>0$, J_r denotes the resolvent of A, i.e., $J_r=(I-rA)^{-1}$.

Consider the resolvent iteration

$$\text{(RI)} \qquad \begin{cases} x_0 \in X \\ x_n = J_{r_n} x_{n-1} \qquad \text{for } n \geqq 1 \end{cases}$$

where $\{r_n\}$ is a sequence of positive numbers. The purpose of this note is to prove the following

Theorem. *$T(t)x$ is strongly* (resp. weakly) *convergent as $t \to \infty$ for all $x \in \overline{D(A)}$ if and only if* (RI) *is strongly* (resp. weakly) *convergent as $n \to \infty$ for all $x_0 \in X$ and all $\{r_n\} \in l^2 \backslash l^1$.*

This theorem has been proved by Passty [1, Theorem 2] under an additional assumption that A is Lipschitzian. We can, however, remove the assumption on A by using the idea of [3].

2. Proof of Theorem. By a *contractive evolution system* on $C(\subset X)$ we mean a two-parameter family $\{U(t, s): 0 \leqq s \leqq t < \infty\}$ of self-maps of C satisfying: (i) $U(t, t)z = z$ for $t \in R^+ = [0, \infty)$ and $z \in C$; (ii) $U(t, s)\ U(s, r)z = U(t, r)z$ for $t \geqq s \geqq r$ in R^+ and $z \in C$; (iii) $\|U(t, s)z_1 - U(t, s)z_2\| \leqq \|z_1 - z_2\|$ for $t \geqq s$ in R^+ and $z_1, z_2 \in C$.

Definition ([1]). A contractive evolution system $\{U(t, s): 0 \leqq s \leqq t < \infty\}$ on $\overline{D(A)}$ is said to be asymptotically equal to the semigroup $\{T(t): t \geqq 0\}$ if for each $x \in \overline{D(A)}$,

(2.1) $\lim_{t\to\infty} \|U(t+h, s)x - T(h)U(t, s)x\| = 0$ for each $s \geqq 0$, uniformly in $h \geqq 0$ and

(2.2) $\lim_{t\to\infty} \|U(t+h, t)T(t)x - T(t+h)x\| = 0$ uniformly in $h \geqq 0$.

The following proposition is due to Passty [1].

Proposition 2.1. *Let $\{U(t, s): 0 \leqq s \leqq t < \infty\}$ be a contractive evolution system which is asymptotically equal to the semigroup $\{T(t): t \geqq 0\}$. Then $T(t)x$ is strongly* (resp. weakly) *convergent as $t \to \infty$ for all $x \in \overline{D(A)}$ if and only if $U(t, s)x$ is strongly* (resp. weakly) *convergent as $t \to \infty$ for all $x \in \overline{D(A)}$ and all $s \geqq 0$.*

Let $\{r_n\}$ be a sequence of positive numbers such that $\{r_n\} \notin l^1$. Put $n(t)=$ "the index n for which $\sum_{i=1}^{n-1} r_i < t \leqq \sum_{i=1}^{n} r_i$" for $t>0$ and $n(0)=0$.

Clearly we have

Lemma 2.2. *Define $U(t, s)$ for $0 \leqq s \leqq t < \infty$ by*

$$U(t, s)x = \prod_{i=n(s)+1}^{n(t)} J_{r_i} x \qquad \text{for } x \in X.$$

Then $\{U(t, s): 0 \leqq s \leqq t < \infty\}$ is a contractive evolution system on X, and $\{U(t, s)|_{\overline{D(A)}}: 0 \leqq s \leqq t < \infty\}$ is a contractive evolution system on $\overline{D(A)}$, where $U(t, s)|_{\overline{D(A)}}$ is the restriction of $U(t, s)$ to $\overline{D(A)}$.

Proposition 2.3. *If $\{r_n\} \in l^2 \backslash l^1$, then $\{U(t, s)|_{\overline{D(A)}}: 0 \leqq s \leqq t < \infty\}$ in Lemma 2.2 is asymptotically equal to the semigroup $\{T(t): t \geqq 0\}$.*

To prove this proposition we use the following

Lemma 2.4. *For $x', z \in X$, $u \in D(A)$, $l \geqq 1$, $i, j \geqq 0$ and $\lambda > 0$ we have*

$$\begin{aligned} \Big\| \prod_{k=l}^{l+i} J_{r_k} x' - J_\lambda^j z \Big\| &\leqq \| J_{r_l} x' - u \| + \| z - u \| \\ &+ \Big\{ \Big(\sum_{k=l+1}^{l+i} r_k - j\lambda\Big)^2 + \sum_{k=l+1}^{l+i} r_k^2 + j\lambda^2 \Big\}^{1/2} \cdot |||Au|||, \end{aligned}$$

where $|||Au||| = \inf\{\|y\| : y \in Au\}$.

Proof. The argument of [2, Lemma 2.1] gives the following estimate:

$$\begin{aligned} (2.3) \qquad \| x_i - \hat{x}_j \| &\leqq \| x_0 - u \| + \| \hat{x}_0 - u \| \\ &+ \Big\{ \Big(\sum_{k=1}^{i} h_k - \sum_{k=1}^{j} \hat{h}_k\Big)^2 + \sum_{k=1}^{i} h_k^2 + \sum_{k=1}^{j} \hat{h}_k^2 \Big\} \cdot |||Au||| \end{aligned}$$

for $x_0, \hat{x}_0 \in X$, $u \in D(A)$, $i, j \geqq 0$ and sequences $\{h_k\}$, $\{\hat{h}_k\}$ of positive numbers, where $x_i = J_{h_i} x_{i-1}$, $\hat{x}_j = J_{\hat{h}_j} \hat{x}_{j-1}$ and $\sum_{k=1}^{0} h_k = \sum_{k=1}^{0} \hat{h}_k = 0$. Let $x', z \in X$, $l \geqq 1$ and $\lambda > 0$. Using (2.3) with $x_0 = J_{r_l} x'$, $\hat{x}_0 = z$ and $h_i = r_{l+i}$, $\hat{h}_j = \lambda$, we obtain the conclusion.

Proof of Proposition 2.3. It suffices to show that (2.1) and (2.2) hold for every $x \in D(A)$. Let $x \in D(A)$.

We first show (2.1). In Lemma 2.4, we let $l = n(t)$, $x' = \prod_{k=n(s)+1}^{n(t)-1} J_{r_k} x$ and $i = n(t+h) - n(t)$, $z = u = U(t, s)x$, and $j = [h/\lambda]$. Then

$$|||AU(t, s)x||| \leqq |||Ax|||$$

implies

$$\begin{aligned} \| U(t+h, s)x - J_\lambda^{[h/\lambda]} U(t, s)x \| \leqq \Big\{ &\sum_{k=n(t)+1}^{n(t+h)} r_k - [h/\lambda]\lambda)^2 \\ &+ \sum_{k=n(t)+1}^{n(t+h)} r_k^2 + [h/\lambda]\lambda^2 \Big\}^{1/2} \cdot |||Ax|||. \end{aligned}$$

As $\lambda \to 0+$,

$$\begin{aligned} \| U(t+h, s)x - T(h)U(t, s)x \| &\leqq \Big\{ \Big(\sum_{k=n(t)+1}^{n(t+h)} r_k - h\Big)^2 + \sum_{k=n(t)+1}^{\infty} r_k^2 \Big\}^{1/2} \cdot |||Ax||| \\ &\leqq \Big(\alpha(t)^2 + \sum_{k=n(t)+1}^{\infty} r_k^2\Big)^{1/2} \cdot |||Ax|||, \end{aligned}$$

where $\alpha(t) = \sup\{r_{n(s)} : s \geqq t\}$. Since $\alpha(t)$ and $\sum_{k=n(t)+1}^{\infty} r_k^2$ are convergent to 0 as $t \to \infty$, we obtain (2.1).

We next show (2.2). In Lemma 2.4, we let $l = n(t)$, $i = n(t+h) - n(t)$, $j = [(t+h)/\lambda] - [t/\lambda]$ and $x' = z = u = J_\lambda^{[t/\lambda]} x$. Then

$$\begin{aligned} &\| U(t+h, t) J_\lambda^{[t/\lambda]} x - J_\lambda^{[(t+h)/\lambda]} x \| \\ &\quad \leqq \| U(t+h, t) J_\lambda^{[t/\lambda]} x - U(t+h, t) J_{r_{n(t)}} J_\lambda^{[t/\lambda]} x \| \\ &\qquad + \| U(t+h, t) J_{r_{n(t)}} J_\lambda^{[t/\lambda]} x - J_\lambda^{[(t+h)/\lambda]} x \| \end{aligned}$$

$$\leqq 2\|J_{r_{n(t)}}J_\lambda^{[t/\lambda]}x - J_\lambda^{[t/\lambda]}x\| + \Big[\Big\{\textstyle\sum_{k=n(t)+1}^{n(t+h)} r_k - \Big(\Big[\frac{t+h}{\lambda}\Big] - \Big[\frac{t}{\lambda}\Big]\Big)\lambda\Big\}^2$$

$$+\textstyle\sum_{k=n(t)+1}^{n(t+h)} r_k^2 + \Big(\Big[\frac{t+h}{\lambda}\Big] - \Big[\frac{t}{\lambda}\Big]\Big)\lambda^2\Big]^{1/2} \cdot |||Ax|||$$

$$\leqq 2r_{n(t)}|||Ax||| + \Big[\Big\{\textstyle\sum_{k=n(t)+1}^{n(t+h)} r_k - \Big(\Big[\frac{t+h}{\lambda}\Big] - \Big[\frac{t}{\lambda}\Big]\Big)\lambda\Big\}^2$$

$$+\textstyle\sum_{k=n(t)+1}^{n(t+h)} r_k^2 + \Big(\Big[\frac{t+h}{\lambda}\Big] - \Big[\frac{t}{\lambda}\Big]\Big)\lambda^2\Big]^{1/2} \cdot |||Ax|||.$$

Here we have used that

$$\|J_{r_{n(t)}}J_\lambda^{[t/\lambda]}x - J_\lambda^{[t/\lambda]}x\| \leqq r_{n(t)}|||AJ_\lambda^{[t/\lambda]}x||| \leqq r_{n(t)}|||Ax|||.$$

Letting $\lambda \to 0+$,

$$\|U(t+h, t)T(t)x - T(t+h)x\|$$
$$\leqq [2r_{n(t)} + \{(\textstyle\sum_{k=n(t)+1}^{n(t+h)} r_k - h)^2 + \sum_{k=n(t)+1}^{\infty} r_k^2\}^{1/2}]\,|||Ax|||$$
$$\leqq \{2r_{n(t)} + (\alpha(t)^2 + \textstyle\sum_{k=n(t)+1}^{\infty} r_k^2)\}^{1/2} \cdot |||Ax|||.$$

So (2.2) holds.

Proof of Theorem. $\Rightarrow$) Let $x_0 \in X$, $\{r_n\} \in l^2\backslash l^1$ and let $U(t, s)$ be as in Lemma 2.2. By virtue of Propositions 2.3 and 2.1, $U(t, 0)x_0 = U(t, r_1)U(r_1, 0)x_0$ is strongly (resp. weakly) convergent as $t \to \infty$. In particular, $x_n = U(\sum_{i=1}^n r_i, 0)x_0$ is strongly (resp. weakly) convergent as $n \to \infty$.

$\Leftarrow$) Let $x \in \overline{D(A)}$, $\{r_n\} \in l^2\backslash l^1$ and let $U(t, s)$ be as in Lemma 2.2. By using Propositions 2.1 and 2.3 again, it suffices to show that $U(t, s)x$ is strongly (resp. weakly) convergent as $t \to \infty$ for every $s \geqq 0$. Now, let $s \geqq 0$ and put $r'_k = r_{k+n(s)}$ for $k = 1, 2, \cdots$. Then

$$U(t, s)x = J_{r'_{n(t)-n(s)}} \cdot \cdots \cdot J_{r'_2} J_{r'_1} x$$

is strongly (resp. weakly) convergent as $t \to \infty$ by our assumption because $\{r'_k\} \in l^2\backslash l^1$.

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