68. A Note on Circumferentially Mean Univalent Functions in an Annulus

By Hitoshi ABE
Department of Applied Mathematics, Faculty of Engineering, Ehime University

(Communicated by Kôsaku YOSIDA, M. J. A., June 14, 1983)

1. **Introduction.** In the previous paper [1] we extended the so-called Montel-Bieberbach's theorem on values omitted by meromorphic and univalent functions in $|z|<1$, to the case of circumferentially mean univalence (defined hereafter). In the next paper [2] we announced the results on meromorphic and circumferentially mean univalent functions in an annulus which mean an extension of the author's results [1]. In this paper, we shall first extend Grötzsch's theorem ([3] or [5]) to the case of circumferentially mean univalence and then prove the author's results [2] in the precise and intrinsic form.

We shall define circumferentially mean univalent functions in a domain D. Let $f(z)$ be regular or meromorphic in D and $n(R, \Phi)$ denote the number of roots of the equation $f(z)=w=Re^{i\Phi}$. We define $p(R)$ as follows.

$$p(R)=\frac{1}{2\pi}\int_0^{2\pi} n(R, \Phi)d\Phi \quad (0\leq R<\infty).$$

If $p(R)\leq 1$ $(0\leq R<\infty)$, $f(z)$ is called "circumferentially mean univalent".

2. We shall first state the following two lemmas.

Lemma 1. *Let $w=f(z)$ be single-valued, regular in $1\leq|z|<R$ and $|f(z)|\leq 1$ there. Moreover let the circle $|z|=1$ be univalently mapped onto the circle $|w|=1$. If we denote the harmonic measure of the circle $|z|=1$ with respect to the annulus $1<|z|<R$ by $\omega(z)$ and do the harmonic measure of $|w|=1$ with respect to the image domain D_f under $w=f(z)$ by $\omega_f(w)$, then we have*

$$(1) \qquad I(\omega(z))\geq I(\omega_f(w)),$$

where $I(\omega(z))$ or $I(\omega_f(w))$ denote the Dirichlet integral of $\omega(z)$ or $\omega_f(w)$ respectively.

Proof. We may consider Landau-Osserman's results [6] by means of exhaustion method.

Lemma 2. *Let $f(z)$ satisfy the same conditions as in Lemma 1 and D_f, or $\omega_f(w)$ denote the same notation in Lemma 1 respectively. If D_f^* denotes the circularly symmetrized domain of D_f with respect to* *the positive real axis and* $\omega_f^*(w)$ *does the harmonic measure of the circle* $|w|=1$ *with respect to* D_f^*, *then we have*

$$(2) \qquad I(\omega_f(w)) \geq I(\omega_f^*(w)).$$

Proof. We may consider quite similarly the method of Haymans' proof of Pólya-Szegö's theorem on circularly symmetrized condenser ([4], [7]).

Now we shall extend Grötzsch's theorem which is an extension of one-quarter theorem.

Theorem 1. *Let* $w=f(z)$ *be single-valued, regular, and circumferentially mean univalent and satisfy the inequality* $|f(z)|\geq 1$ *in* $1\leq |z|<R$. *If the circle* $|z|=1$ *is mapped onto the circle* $|w|=1$, *then the image domain* D_f *under* $w=f(z)$ *always covers the annulus* $1\leq |w| < P^*$ $(P^*\geq P)$ *where* P *is determined by the relation* $\Phi(P)=R$ *with respect to Grötzsch extremal domain* ([3] *or* [5]). $P^*=P$ *occurs when* $f(z)$ *maps univalently the annulus* $1<|z|<R$ *onto Grötzsch extremal domain.*

Proof. We consider $g(z)=1/f(z)$. $g(z)$ is single-valued, regular and circumferentially mean univalent in $1\leq |z|<R$ and $|g(z)|\leq 1$ there. Moreover we see the univalency of $g(z)$ on the circle $|z|=1$ by means of the definition of circumferentially mean univalence. Here let D_g be the image domain of the annulus $1\leq |z|<R$ under $w=g(z)$ and D_g^* be the circularly symmetrized domain of D_g with respect to the positive real axis. Then the complementary set E_g of D_g with respect to the unit circle $|w|\leq 1$, is transformed to the circularly symmetrized set E_g^*. Now we prove that the intersection S of E_g^* and the positive real axis consists of only one interval $[o, Q]$ where we put $Q=\mathrm{Max}|w_c|$ $(w_c \in E_g)$. Suppose $r \notin S$ where $o<r<Q$. Then the circle $|w|=r$ is wholly contained in D_g. Moreover we see by means of the circumferentially mean univalence of $g(z)$ that the circle $|w|=r$ is the univalent image of a Jordan curve C in the annulus $1\leq |z|<R$.

(i) If the domain enclosed by C is wholly contained in the annulus $1<|z|<R$, we see by means of Darboux's theorem that the circle $|w|\leq r$ is wholly contained in D_g. This is absurd.

(ii) If C encloses the circle $|z|=1$, we see also by means of the slight extension of Darboux's theorem that the annulus $r<|w|<1$ corresponds univalently to the ring domain enclosed by the circle $|z|=1$ and C. This is also absurd.

Now let D_0 be the unit circle $|w|\leq 1$ with the slit $[o, Q]$. Then $D_0 \supset D_g^*$. Here let $M(D_g^*)$ or $M(D_0)$ denote Modul of D_g^* or D_0 respectively. Then by means of Lemmas 1 and 2, we have the following relation

$$(3) \qquad \log R \leq M(D_g^*) \leq M(D_0),$$

because Dirichlet integral of harmonic measure equals $2\pi\times$(reciprocal of Modul of ring domain).

On the other hand let D_f^* be the circularly symmetrized domain of D_f with respect to the positive real axis and D_0' be the outer circle $|w|\geq 1$ with the slit $[1/Q, \infty]$. Then the intersection of the complementary set E_f^* of D_f^* with respect to the outer circle $|w|>1$ and the positive real axis is the slit $[1/Q, \infty]$ and Modul of D_0' equals Modul of D_0. Therefore $P\leq P^*(1/Q=P^*)$. This completes the proof.

As an application of Theorem 1 we have the following which is an extension of the author's results [1].

Theorem 2. *Let $w=f(z)$ be meromorphic and circumferentially mean univalent in the annulus $1\leq|z|<R$ and satisfy the inequality $|f(z)|\geq 1$ there. Moreover let the circle $|z|=1$ be mapped onto the circle $|w|=1$. If E_f denotes the complementary set of the image domain D_f under $w=f(z)$ with respect to the circle $|w|>1$ and we put $\alpha=\text{Min}\,|w_c|$, $\beta=\text{Max}\,|w_c|$ where $w_c\in E_f$, then Modul $M(\alpha,\beta)$ of the unit circle $|w|>1$ with the slit $[\alpha,\beta]$ satisfies the following inequality.*

(4) $$M(\alpha,\beta)\geq\log R.$$

Accordingly we have the following inequality.

(5) $$\alpha\geq\frac{\beta P+1}{P+\beta},\ \textit{that is},\ \beta\leq\frac{\alpha P-1}{P-\alpha},$$

where P is defined in Theorem 1.

Proof. By means of considering $w=1/f(z)$, the relation (4) can be derived quite similarly as in the proof of Theorem 1. Next by the linear transformation $(1-\beta w)/(w-\beta)$, the circle $|w|>1$ with the slit $[\alpha,\beta]$ is transformed to the circle $|w|>1$ with the slit $[1-\alpha\beta/\alpha-\beta, \infty]$. Hence by means of Theorem 1 and (4) we have

(6) $$P\leq\frac{1-\alpha\beta}{\alpha-\beta}.$$

From this (5) is directly derived.

Remark. The results in Theorems 1 and 2 can be extended to the case of circumferentially mean p valence.

References

[1] H. Abe: On meromorphic and circumferentially mean univalent functions. J. Math. Soc. Japan, **16**, 342–351 (1964).
[2] ——: On some analytic functions in an annulus. Mem. Ehime Univ., Sec. III, **8**, 269–272 (1976).
[3] H. Grötzsch: Über einige Extremalprobleme der Korformen Abbildung I, II. Leipziger Berichte, **80**, 367–376; 497–502 (1928).
[4] W. K. Hayman: Multivalent functions. Cambridge Tracts, no. 48 (1958).
[5] Y. Komatu: Conformal Mapping II. Kyoritu Tracts (Tokyo) (1949) (in Japanese).

[6] H. J. Landau and R. Osserman: Some distortion theorems for multivalent mappings. Proc. Amer. Math. Soc., **10**, 87–91 (1959).
[7] G. Pólya: Sur la symétrisation circulaire. C. R. Acad. Sci. Paris, **230**, 25–27 (1950).