32. On Certain Cubic Fields. II

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1. For an algebraic number field F, we denote the class number and the regulator of F by h_F and R_F respectively. The notations E_F and D_F have the same meanings as in [4].

In [2], C. L. Siegel proved the following:

Theorem. *Let F be imaginary quadratic field with discriminant D_F. If $|D_F| \to \infty$, then $h_F \to \infty$.*

To prove this, Siegel used the formula

$$(*) \qquad \lim_{D_F \to \infty} \frac{\log h_F R_F}{\log \sqrt{|D_F|}} = 1,$$

which was established first by Siegel [2] for quadratic fields F, then by Brauer [1] for general algebraic number fields F.

The purpose of this note is to show that an analogous result holds for the class of cubic fields treated in [4].

We shall prove:

Theorem. *Let K be cyclic cubic field $K = \boldsymbol{Q}(\theta)$,* Irr $(\theta : \boldsymbol{Q}) = f(x) = x^3 - mx^2 - (m+3)x - 1$, $m \in \boldsymbol{Z}$, *with square free m^2+3m+9. Then $h_K \to \infty$ as $D_K \to \infty$.*

Remark. There are infinitely many rational integers m such that m^2+3m+9 is square free (cf. [3]).

2. *Proof of Theorem.* We see easily that the roots of $f(x)$ can be denoted by $\theta, \theta', \theta''$ so that they satisfy

$$-\frac{m+1}{m} < \theta < -\frac{m^2+1}{m^2}, \quad -\frac{1}{m} < \theta' < -\frac{1}{m^2} \quad \text{and} \quad m+1 < \theta'' < m+2$$

when $m \geqq 1$.

In [4], we have proved that $E_K = \langle \pm 1 \rangle \times \langle \theta, \theta' \rangle$. So we get $0 < R_K = \text{abs}\,\{\log|\theta| \log|\theta''| - (\log|\theta'|)^2\} < \log(m+1)\log(m+2)$ because $\log|\theta| < \log(m+1)$, $\log|\theta''| < \log(m+2)$. This yields

$$R_K = o(m^2) = o(\sqrt{D_K})$$

as we have $\sqrt{D_K} = m^2+3m+9$ because m^2+3m+9 is square free.

Now, the formula $(*)$ holds for the set of all algebraic number fields F. So it holds also for our class of cubic fields K, where $\sqrt{D_K} = m^2+3m+9$, and $D_K \to \infty$ means the same meaning as $m \to \infty$.

Therefore we obtain

$$1=\lim_{D_K\to\infty}\frac{\log h_K R_K}{\log\sqrt{D_K}}=\lim_{m\to\infty}\frac{\log R_K}{\log(m^2+3m+9)}+\lim_{m\to\infty}\frac{\log h_K}{\log(m^2+3m+9)}$$
$$=\lim_{m\to\infty}\frac{\log h_K}{\log(m^2+3m+9)}.$$

Thus we have $h_K\to\infty$ as $D_K\to\infty$. This completes the proof of Theorem.

References

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