94. *Generic Bifurcations of Varieties*[*)]

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Let $f:(\boldsymbol{R}^n\times\boldsymbol{R}^r,0)\to(\boldsymbol{R}^p,0)$ be a smooth map germ. For each $u\in(\boldsymbol{R}^r,0)$, we have a germ of "varieties" $f_u^{-1}(0)$ defined by $f_u=f\,|\,\boldsymbol{R}^r\times u$. In this note, we shall announce some results about bifurcations of $f_u^{-1}(0)$ as u varies in $(\boldsymbol{R}^r,0)$. Details will appear elsewhere.

1. Parametrised contact equivalence. The local ring $C_0^\infty(\boldsymbol{R}^n\times\boldsymbol{R}^r)$ is the ring of smooth function germs $(\boldsymbol{R}^n\times\boldsymbol{R}^r,0)\to\boldsymbol{R}$. This ring has a maximal ideal $\mathfrak{M}_{n+r}$ consisting of all germs with $f(0)=0$. For a smooth map germ $f:(\boldsymbol{R}^n\times\boldsymbol{R}^r,0)\to(\boldsymbol{R}^p,0)$, we denote $I(f)=f^*(\mathfrak{M}_p)C_0^\infty(\boldsymbol{R}^n\times\boldsymbol{R}^r)$, where $f^*:C_0^\infty(\boldsymbol{R}^p)\to C_0^\infty(\boldsymbol{R}^n\times\boldsymbol{R}^r)$ is defined by $f^*(h)=h\circ f$.

Definition 1. Map germs $f,\ g:(\boldsymbol{R}^n\times\boldsymbol{R}^r,0)\to(\boldsymbol{R}^p,0)$ are *P-$\mathcal{K}$-equivalent* (resp. *$S.P$-$\mathcal{K}$-equivalent*) if there exists a diffeomorphism germ on $(\boldsymbol{R}^n\times\boldsymbol{R}^r,0)$ of the form $\Phi(x,u)=(\Phi_1(x,u),\phi(u))$ (resp. $\Phi(x,u)=(\Phi_1(x,u),u)$) such that $\Phi^*(I(f))=I(g)$. We denote $f\sim_{P\text{-}\mathcal{K}}g$ (resp. $f\sim_{S.P\text{-}\mathcal{K}}g$).

For each smooth map germ $f:(\boldsymbol{R}^n\times\boldsymbol{R}^r,0)\to(\boldsymbol{R}^p,0)$, the *bifurcation map germ* $\pi_f:(f^{-1}(0),0)\to(\boldsymbol{R}^r,0)$ is defined by $\pi_f(x,u)=u$.

Definition 2. For two map germs $f,g:(\boldsymbol{R}^n\times\boldsymbol{R}^r,0)\to(\boldsymbol{R}^p,0)$, bifurcation map germs π_f,π_g are *$\mathcal{A}$-equivalent* if there are diffeomorphism germs Φ on $(\boldsymbol{R}^n\times\boldsymbol{R}^r,0)$ and ϕ on $(\boldsymbol{R}^r,0)$ such that $\Phi(f^{-1}(0))=g^{-1}(0)$ and $\phi\circ\pi_f=\pi_g\circ\Phi$.

Remarks. i) If f,g are P-$\mathcal{K}$-equivalent, then $\pi_f,\ \pi_g$ are $\mathcal{A}$-equivalent.

ii) For each $f:(\boldsymbol{R}^n,0)\to(\boldsymbol{R}^p,0)$, we define $D_f:(\boldsymbol{R}^n\times\boldsymbol{R}^p,0)\to(\boldsymbol{R}^p,0)$ by $D_f(x,y)=f(x)-y$. We can see that P-$\mathcal{K}$-equivalence theory is one of the generalization of Mather's $\mathcal{A}$-equivalence theory (cf. [3], [4]).

iii) The case when $r=1$, this equivalence relation has been studied by M. Golubitsky and D. Schaeffer ([1]). But the situation is quite different in the case of $r\geq 2$ (see the next section).

2. Finite determinacy. Definition 3. Let $f,g:(\boldsymbol{R}^n\times\boldsymbol{R}^r,0)\to(\boldsymbol{R}^p,0)$ be smooth map germs. i) f,g are *k-jet equivalent* if $(f^*-g^*)(\mathfrak{M}_p)\subset\mathfrak{M}_{n+r}^{k+1}$. ii) f,g are *(k_1,k_2)-jet equivalent* if $(f^*-g^*)(\mathfrak{M}_p)\subset(\mathfrak{M}_n^{k_1+1}+\mathfrak{M}_r^{k_2+1})C_0^\infty(\boldsymbol{R}^n\times\boldsymbol{R}^r)$.

These are clearly equivalence relations; we respectively denote j_0^kf and $j_0^{(k_1,k_2)}f$ of equivalence classes represented by f.

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Definition 4. i) Map germ $f:(\boldsymbol{R}^n\times\boldsymbol{R}^r,0)\to(\boldsymbol{R}^p,0)$ is *k-determined* (resp. *(k_1,k_2)-determined*) *relative to* $\mathcal{S}$ if every map germ $g:(\boldsymbol{R}^n\times\boldsymbol{R}^r,0)\to(\boldsymbol{R}^p,0)$ such that $j_0^kf=j_0^kg$ (resp. $j_0^{(k_1,k_2)}f=j_0^{(k_1,k_2)}g$) is $\mathcal{S}$-equivalent to f. ii) Map germ $f:(\boldsymbol{R}^n\times\boldsymbol{R}^r,0)\to(\boldsymbol{R}^p,0)$ is *finitely determined* (resp. *separated finitely determined*) *relative to* $\mathcal{S}$ if there exists $k\in\boldsymbol{N}$ (resp. $(k_1,k_2)\in\boldsymbol{N}\times\boldsymbol{N}$) such that f is k-determined (resp. (k_1,k_2)-determined) relative to $\mathcal{S}$. Where $\mathcal{S}$ is P-$\mathcal{K}$ or $S.P$-$\mathcal{K}$.

Let $C_0^\infty(\boldsymbol{R}^n\times\boldsymbol{R}^r,\boldsymbol{R}^p)$ be the $C_0^\infty(\boldsymbol{R}^n\times\boldsymbol{R}^r)$-module of smooth map germs $(\boldsymbol{R}^n\times\boldsymbol{R}^r,0)\to\boldsymbol{R}^p$. For each germ $f:(\boldsymbol{R}^n\times\boldsymbol{R}^r,0)\to(\boldsymbol{R}^p,0)$, we denote

$$T_e(P\text{-}\mathcal{K})(f)=\left\langle\frac{\partial f}{\partial x_1},\cdots,\frac{\partial f}{\partial x_n}\right\rangle C_0^\infty(\boldsymbol{R}^n\times\boldsymbol{R}^r)+\left\langle\frac{\partial f}{\partial u_1},\cdots,\frac{\partial f}{\partial u_r}\right\rangle C_0^\infty(\boldsymbol{R}^r)$$
$$+f^*(\mathfrak{M}_p)C_0^\infty(\boldsymbol{R}^n\times\boldsymbol{R}^r,\boldsymbol{R}^p),$$
$$T_e(S.P\text{-}\mathcal{K})(f)=\left\langle\frac{\partial f}{\partial x_1},\cdots,\frac{\partial f}{\partial x_n}\right\rangle C_0^\infty(\boldsymbol{R}^n\times\boldsymbol{R}^r)+f^*(\mathfrak{M}_p)C_0^\infty(\boldsymbol{R}^n\times\boldsymbol{R}^r,\boldsymbol{R}^p)$$

and

$$T(S.P\text{-}\mathcal{K})(f)=\mathfrak{M}_{n+r}\left\langle\frac{\partial f}{\partial x_1},\cdots,\frac{\partial f}{\partial x_n}\right\rangle C_0^\infty(\boldsymbol{R}^n\times\boldsymbol{R}^p)$$
$$+f^*(\mathfrak{M}_p)C_0^\infty(\boldsymbol{R}^n\times\boldsymbol{R}^r,\boldsymbol{R}^p).$$

Then we have the following theorems.

Theorem A (Characterization theorem). *The following are equivalent.*

1) *f is finitely determined relative to P-$\mathcal{K}$* (resp. $S.P$-$\mathcal{K}$).
2) *f is separated finitely determined relative to P-$\mathcal{K}$* (resp. $S.P$-$\mathcal{K}$).
3) *For some integer k, $\mathfrak{M}_{n+r}^kC_0^\infty(\boldsymbol{R}^n\times\boldsymbol{R}^r,\boldsymbol{R}^p)\subset T_e(P\text{-}\mathcal{K})(f)$* (resp. $T_e(S.P\text{-}\mathcal{K})(f)$).
4) $\dim_{\boldsymbol{R}} C_0^\infty(\boldsymbol{R}^n\times\boldsymbol{R}^r,\boldsymbol{R}^p)/T_e(P\text{-}\mathcal{K})(f)<+\infty$ (resp. $\dim_{\boldsymbol{R}} C_0^\infty(\boldsymbol{R}^n\times\boldsymbol{R}^r,\boldsymbol{R}^p)/T_e(S.P\text{-}\mathcal{K})(f)<+\infty$).

Theorem B. *Let $f:(\boldsymbol{R}^n\times\boldsymbol{R}^r,0)\to(\boldsymbol{R}^p,0)$ be a smooth map germ.*

i) *Let D be a $C_0^\infty(\boldsymbol{R}^n\times\boldsymbol{R}^r)$-submodule of $C_0^\infty(\boldsymbol{R}^n\times\boldsymbol{R}^r,\boldsymbol{R}^p)$. If $D\subset T_e(P\text{-}\mathcal{K})(f)+(\mathfrak{M}_n^{s_1}+\mathfrak{M}_r^{s_2})C_0^\infty(\boldsymbol{R}^n\times\boldsymbol{R}^r,\boldsymbol{R}^p)$ and*

$$(\mathfrak{M}_n^{s_1}+\mathfrak{M}_r^{s_2})C_0^\infty(\boldsymbol{R}^n\times\boldsymbol{R}^r,\boldsymbol{R}^p)\subset T(S.P\text{-}\mathcal{K})(f)+\mathfrak{M}_rD$$
$$+\mathfrak{M}_{n+r}(\mathfrak{M}_n^{s_1}+\mathfrak{M}_r^{s_2})C_0^\infty(\boldsymbol{R}^n\times\boldsymbol{R}^r,\boldsymbol{R}^p)$$

then f is (s_1,s_2)-determined relative to P-$\mathcal{K}$.

ii) *If $r=1$ and $(\mathfrak{M}_n^{s_1}+\mathfrak{M}_1^{s_2})C_0^\infty(\boldsymbol{R}^n\times\boldsymbol{R},\boldsymbol{R}^p)\subset T(S.P\text{-}\mathcal{K})(f)+\mathfrak{M}_{n+1}(\mathfrak{M}_n^{s_1}+\mathfrak{M}_1^{s_2})C_0^\infty(\boldsymbol{R}^n\times\boldsymbol{R},\boldsymbol{R}^p)$ then f is (s_1,s_2)-determined relative to $S.P$-$\mathcal{K}$.*

Remarks. 1) In [1], there is the following estimate: If $\mathfrak{M}_{n+1}^kC_0^\infty(\boldsymbol{R}^n\times\boldsymbol{R},\boldsymbol{R}^p)\subset T(S.P\text{-}\mathcal{K})(f)$, then f is k-determined relative to P-$\mathcal{K}$. The statement of ii) is better than their estimate.

2) We have many other estimates as corollaries of the above theorem. For example, if $C_0^\infty(\boldsymbol{R}^n\times\boldsymbol{R}^r,\boldsymbol{R}^p)=T_e(P\text{-}\mathcal{K})(f)$, then f is $(r+1,1)$-determined relative to P-$\mathcal{K}$. This is a generalization of

Mather's theorem (cf. [4], Proposition 3.5).

In the case where $r \geqq 2$, situations are more complicated as follows.

Proposition C. *Let* $f:(\boldsymbol{R}^n \times \boldsymbol{R}^r, 0) \to (\boldsymbol{R}^p, 0)$ *be a smooth map germ with* $r \geqq 2$. *The following are equivalent.*

1) f *is finitely determined relative to S.P-*$\mathcal{K}$.

2) f *is S.P-*$\mathcal{K}$*-equivalent to the following germ;*

$$(x_1, \cdots, x_n, u_1, \cdots, u_r) \longmapsto (x_1, \cdots, x_p).$$

3. **Versal deformations.** Definitions of the deformation of a smooth map germ and its versality with respect to P-$\mathcal{K}$-equivalence is analogous to those of smooth section germs in [2]. Then we have the versality theorem for P-$\mathcal{K}$-equivalence.

Theorem D. *Let* $F:(\boldsymbol{R}^n \times \boldsymbol{R}^r \times \boldsymbol{R}^s, 0) \to (\boldsymbol{R}^p, 0)$ *be a s-parameter deformation of* $f:(\boldsymbol{R}^n \times \boldsymbol{R}^r, 0) \to (\boldsymbol{R}^p, 0)$. *If* $C_0^\infty(\boldsymbol{R}^n \times \boldsymbol{R}^r, \boldsymbol{R}^p) = T_e(P\text{-}\mathcal{K})(f) + \langle \partial F/\partial x_1 | \boldsymbol{R}^n \times 0, \cdots, \partial F/\partial x_n | \boldsymbol{R}^n \times 0, e_1, \cdots, e_p \rangle_{\boldsymbol{R}}$, *then* F *is a P-*$\mathcal{K}$*-versal deformation of* f. *Here,* $e_i = (0, \cdots, 0, 1, 0, \cdots, 0)$.

Remark. The above theorem is not a corollary of Theorem B in [2]. For the proof, we must use a generalization of the preparation theorem in ([5], Corollary 1.7).

4. **Classifications.** For each $f:(\boldsymbol{R}^n \times \boldsymbol{R}^r, 0) \to (\boldsymbol{R}^p, 0)$, we define $df_x: T_0\boldsymbol{R}^n \times \boldsymbol{R}^r \to T_0\boldsymbol{R}^p$ by $df_x(v_i, w_j) = (\sum_{i=1}^n v_i(\partial f_l/\partial x_i))$, $df_u: T_0\boldsymbol{R}^n \times \boldsymbol{R}^r \to T_0\boldsymbol{R}^p$ by $df_u(v_i, w_j) = (\sum_{j=1}^r w_j(\partial f_l/\partial u_j))$, and $df_u^K = \pi \circ df_u | \operatorname{Ker} df_x$: $\operatorname{Ker} df_x \to \operatorname{Coker} df_x$, where $\pi: T_0\boldsymbol{R}^p \to \operatorname{Coker} df_x$ is the canonical projection.

Definition 4. i) We say that f has the $\sum_s^k$*-type at* 0 if $\operatorname{rank} df_x = \min(n, p) - k$ and $\operatorname{rank} df_u^K = \min(r, p - \operatorname{rank} df_x) - s$.

ii) We say that f is *non-singular at* 0 if f has the $\sum_0^0$-type at 0.

The following is the implicit function theorem relative to P-$\mathcal{K}$.

Theorem E. *Let* $f:(\boldsymbol{R}^n \times \boldsymbol{R}^r, 0) \to (\boldsymbol{R}^p, 0)$ *be a non-singular germ.*

1) *If* $n \geqq p$, *then* f *is S.P-*$\mathcal{K}$*-equivalent to the following germ;* $(x_1, \cdots, x_n, u_1, \cdots, u_r) \longmapsto (x_1, \cdots, x_p)$.

2) *If* $n < p$ *and* $r \leqq p - n$, *then* f *is P-*$\mathcal{K}$*-equivalent to the following germ;* $(x_1, \cdots, x_n, u_1, \cdots, u_r) \longmapsto (x_1, \cdots, x_n, u_1, \cdots, u_r, 0, \cdots, 0)$.

3) *If* $n < p$ *and* $r > p - n$, *then* f *is P-*$\mathcal{K}$*-equivalent to the following germ;* $(x_1, \cdots, x_n, u_1, \cdots, u_r) \longmapsto (x_1, \cdots, x_n, u_1, \cdots, u_{p-n})$.

We now set P-$\mathcal{K}$-codim $(f) = \dim_{\boldsymbol{R}} C_0^\infty(\boldsymbol{R}^n \times \boldsymbol{R}^r, \boldsymbol{R}^p)/T_e(P\text{-}\mathcal{K})(f)$.

Theorem F (Classification theorem in the case of P-$\mathcal{K}$-codimension $=0$). *Let* $f:(\boldsymbol{R}^n \times \boldsymbol{R}^r, 0) \to (\boldsymbol{R}^p, 0)$ *be a smooth map germ with* P-$\mathcal{K}$-codim $(f) = 0$. *If* f *has the* $\sum_{\min(r, p-s)-q}^{\min(n,p)-s}$-type *at* 0, *then* $p = s + q$ *and there exist smooth map germs* $g, \xi_1, \cdots, \xi_{r-q}: (\boldsymbol{R}^{n-s}, 0) \to (\boldsymbol{R}^q, 0)$ *with* $\operatorname{rank} dg = 0$ *and*

$$C_0^\infty(\boldsymbol{R}^{n-s}, \boldsymbol{R}^q) \Big/ \left\langle \frac{\partial g}{\partial x_{s+1}}, \cdots, \frac{\partial g}{\partial x_n} \right\rangle C_0^\infty(\boldsymbol{R}^{n-s}) + g^*(\mathfrak{M}_q) C_0^\infty(\boldsymbol{R}^{n-s}, \boldsymbol{R}^q)$$
$$= \langle \xi_1, \cdots, \xi_{r-q}, e_1, \cdots, e_q \rangle_{\boldsymbol{R}}$$

such that f is P-$\mathcal{K}$-equivalent to the following germ;

$$(x_1, \cdots, x_n, u_1, \cdots, u_r) \longmapsto (x_1, \cdots, x_s, u_1+g_1(x_{s+1}, \cdots, x_n) + u_{q+1}\xi_1^1(x_{s+1}, \cdots, x_n) + \cdots + u_r\xi_{r-q}^1(x_{s+1}, \cdots, x_n), \cdots, u_q + g_q(x_{s+1}, \cdots, x_n) + u_{q+1}\xi_1^q(x_{s+1}, \cdots, x_n) + \cdots + u_r\xi_{r-q}^q(x_{s+1}, \cdots, x_n)).$$

Here, $g=(g_1, \cdots, g_q)$ *and* $\xi_j=(\xi_j^1, \cdots, \xi_j^q)$ *for any* $j=1, \cdots, r-q$.

Remark. In the above theorem, g is the $\mathcal{K}$-finite map in the sense of Mather.

In the case of positive codimensions, we have the following

Theorem G. *Let* $f:(\boldsymbol{R}^n\times \boldsymbol{R}, 0)\to(\boldsymbol{R}^p, 0)$ *be a smooth map germ with* $n\geqq p$ *and* P-$\mathcal{K}$-codim $(f)\leqq 4$.

1) *If f has the* $\sum_0^1$*-type at* 0, *it is P-$\mathcal{K}$-equivalent to one of the following germs:*

P-$\mathcal{K}$-codim (f)	Normal forms
0	$(x_1, \cdots, x_{p-1}, u+Q(x_p, \cdots, x_n))$
1	$(x_1, \cdots, x_{p-1}, u+x_p^3+Q(x_{p+1}, \cdots, x_n))$
2	$(x_1, \cdots, x_{p-1}, u\pm x_p^4+Q(x_{p+1}, \cdots, x_n))$
3	$(x_1, \cdots, x_{p-1}, u+x_p^5+Q(x_{p+1}, \cdots, x_n))$
	$(x_1, \cdots, x_{p-1}, u+x_p^3+x_{p+1}^3+Q(x_{p+2}, \cdots, x_n))$
	$(x_1, \cdots, x_{p-1}, u+x_p^3-x_px_{p+1}^2+Q(x_{p+2}, \cdots, x_n))$
4	$(x_1, \cdots, x_{p-1}, u\pm x_p^6+Q(x_{p+1}, \cdots, x_n))$
	$(x_1, \cdots, x_{p-1}, u\pm(x_p^2x_{p+1}+x_{p+1}^4)+Q(x_{p+2}, \cdots, x_n))$

2) *If f has the* $\sum_1^1$*-type at* 0, *then it is P-$\mathcal{K}$-equivalent to one of the following germs:*

P-$\mathcal{K}$-codim (f)	Normal forms
1	$(x_1, \cdots, x_{p-1}, \pm u^2+Q(x_p, \cdots, x_n))$
2	$(x_1, \cdots, x_{p-1}, u^3+Q(x_p, \cdots, x_n))$
	$(x_1, \cdots, x_{p-1}, \pm u^2+x_p^3+Q(x_{p+1}, \cdots, x_n))$
	$(x_1, \cdots, x_{p-1}, x_p^3\pm ux_p+Q(x_{p+1}, \cdots, x_n))$
3	$(x_1, \cdots, x_{p-1}, \pm u^4+Q(x_p, \cdots, x_n))$
	$(x_1, \cdots, x_{p-1}, x_p^4\pm ux_p+Q(x_{p+1}, \cdots, x_n))$
4	$(x_1, \cdots, x_{p-1}, u^5+Q(x_p, \cdots, x_n))$
	$(x_1, \cdots, x_{p-1}, x_p^5\pm ux_p+Q(x_{p+1}, \cdots, x_n))$

Here, $Q(x_i, \cdots, x_n)=\pm x_i^2\pm\cdots\pm x_n^2$.

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