9. On Eisenstein Series of Degree Two for Hilbert-Siegel Modular Groups

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Introduction. In this note we present an explicit formula for Fourier coefficients of generalized Eisenstein series of degree two for Hilbert-Siegel modular groups in the sense of Langlands [8] and Klingen [4]. This explicit formula is a generalization of the previous result in [7] [11] (the Siegel modular case), and has an application to the algebraicity of the special value of the "second" L-function attached to a Hilbert modular form. Details will appear elsewhere. The author would like to thank Prof. N. Kurokawa for suggestions and encouragements.

§ 1. Generalized Eisenstein series for Hilbert-Siegel modular groups. Let F be a totally real number field of degree g over $\boldsymbol{Q}$, $\mathcal{O}_F$ the ring of integers in F, $E=\mathcal{O}_F^\times$ the group of units in F, and $E_+=\{\varepsilon\in E\,|\,\varepsilon\gg 0\}$ the group of totally positive units in F. Let $F^{(1)},\cdots,F^{(g)}$ be the conjugates of F over $\boldsymbol{Q}$ with $F^{(1)}=F$. The image of an element $\lambda\in F$ (resp. a matrix M with all entries lying in F) under $F\to F^{(i)}$ is denoted by $\lambda^{(i)}$ (resp. $M^{(i)}$). If $\lambda^{(i)}>0$ (resp. ${}^tM=M$ and $M^{(i)}>0$; ${}^tM=M$ and $M^{(i)}\geqq 0$) for $1\leqq i\leqq g$, we write $\lambda\gg 0$ (resp. $M\gg 0$; $M\ggg 0$). For a positive integer n, we put $\Gamma_F^{(n)}=\{M\in M_{2n}(\mathcal{O}_F)\,|\,{}^tMJ_nM=\varepsilon J_n$ for some $\varepsilon\in E_+\}$ where $J_n=\begin{pmatrix}0 & E_n\\ -E_n & 0\end{pmatrix}$ (E_n is the identity matrix of size n). For an integer $k\geqq 0$, we denote by $M_k(\Gamma_F^{(n)})$ (resp. $S_k(\Gamma_F^{(n)})$) the $\boldsymbol{C}$-vector space of all Hilbert-Siegel modular (resp. cusp) forms of weight k with respect to $\Gamma_F^{(n)}$. We denote by $E_k(\Gamma_F^{(n)})$ the orthogonal complement of $S_k(\Gamma_F^{(n)})$ in $M_k(\Gamma_F^{(n)})$ with respect to the Petersson inner product. As usual we put $M_k(\Gamma_F^{(0)})=S_k(\Gamma_F^{(0)})=\boldsymbol{C}$.

Let n, k, r be integers such that $n\geqq 1$, $0\leqq r\leqq n$, $k>n+r+1$. We assume the following condition:

(a) k is an even integer if F contains a unit with norm -1.

Generalizing the construction of Klingen [4], we put

$$E_{n,r}^k(Z,f)=\sum_{M:\Delta_{n,r}\backslash\Gamma_F^{(n)}} f(M\langle Z\rangle^*)\mathrm{N}_{F/Q}(|CZ+D|)^{-k}\qquad \text{for } f\in S_k(\Gamma_F^{(r)}).$$

Here $\Delta_{n,r}$ is the subgroup of $\Gamma_F^{(n)}$ of all $M\in\Gamma_F^{(n)}$ whose entries in the first $n+r$ columns and last $n-r$ rows vanish, and $M=\begin{pmatrix}A & B\\ C & D\end{pmatrix}$ (A,B,C,D are square matrices of size n) runs over a complete system of representatives of the left cosets of $\Gamma_F^{(n)}$ modulo $\Delta_{n,r}$; $Z=(Z_1,\cdots,Z_g)$ is a variable on $\mathfrak{H}_n^g$, the product of g-copies of the Siegel upper half space of degree n; for each $M=\begin{pmatrix} A & B \\ C & D\end{pmatrix}\in\Gamma_F^{(n)}$ we put $M\langle Z\rangle^*=(M^{(1)}\langle Z_1\rangle^*,\cdots,M^{(g)}\langle Z_g\rangle^*)$ where $M^{(i)}\langle Z_i\rangle=(A^{(i)}Z_i+B^{(i)})(C^{(i)}Z_i+D^{(i)})^{-1}$ and $M^{(i)}\langle Z_i\rangle^*$ is the square matrix formed by the first (r,r)-entries of $M^{(i)}\langle Z_i\rangle$; and $\mathrm{N}_{F/Q}(|CZ+D|)=\prod_{1\leq i\leq g}|C^{(i)}Z_i+D^{(i)}|$, $|\ \ |$ denoting the determinant. Then the above summation defining $E_{n,r}^k(Z,f)$ converges uniformly and absolutely on $\{(Z_1,\cdots,Z_g)\in\mathfrak{H}_n^g|\ \mathrm{trace}\,(X_i^2)\leqq c^{-1},\ Y_i\geqq cE_n\ (1\leqq i\leqq g)\}$ $(Z_i=X_i+\sqrt{-1}\,Y_i$, X_i and Y_i real matrices) for any $c>0$ and represents an element of $M_k(\Gamma_F^{(n)})$. Moreover $\Phi E_{n,r}^k(*,f)=E_{n-1,r}^k(*,f)$ for $r<n$ and $\Phi E_{n,n}^k(*,f)=\Phi f=0$ where Φ is the Siegel operator. (For definitions, we refer to Christian [1].) Results in Klingen [4] are generalized to the Hilbert- Siegel case as follows:

Proposition. *Let n, k be integers such that $k>2n\geqq 0$, and suppose that k satisfies the condition* (a). *Put $E_k^r(\Gamma_F^{(n)})=\{E_{n,r}^k(*,f)\,|\,f\in S_k(\Gamma_F^{(r)})\}$ for $0\leqq r\leqq n$. Then:*

(1) *$M_k(\Gamma_F^{(n)})=\oplus_{0\leqq r\leqq n}E_k^r(\Gamma_F^{(n)})$, $E_k(\Gamma_F^{(n)})=\oplus_{0\leqq r\leqq n-1}E_k^r(\Gamma_F^{(n)})$, and $S_k(\Gamma_F^{(n)})=E_k^n(\Gamma_F^{(n)})$.*

(2) *Φ induces a $\boldsymbol{C}$-linear isomorphism: $E_k^r(\Gamma_F^{(n)})\rightleftarrows E_k^r(\Gamma_F^{(n-1)})$ for each $r=0,\cdots,n-1$ if $n\geqq 1$.*

§2. An explicit formula of Fourier coefficients for degree two case. Throughout this section we assume the following conditions: (b) F is a totally real number field with the class number one in the narrow sense, and (c) the rational prime 2 decomposes completely in F. Let $k>0$ be an integer satisfying the condition (a) in §1. Let $f\in M_k(\Gamma_F^{(1)})$ be a normalized eigen Hilbert modular form in the following sense: $f(z)=\sum_{0\preccurlyeq\lambda\in\mathfrak{d}^{-1}}a((\lambda)\mathfrak{d})\boldsymbol{e}(\mathrm{tr}_{F/Q}(\lambda z))$ with $a(\mathcal{O}_F)=1$ and $T(\mathfrak{m})f=a(\mathfrak{m})f$ for all Hecke operators $T(\mathfrak{m})$ associated with integral ideals $\mathfrak{m}$ of F. Here, $\boldsymbol{e}(x)=\exp(2\pi\sqrt{-1}x)$, $z=(z_1,\cdots,z_g)\in\mathfrak{H}_1^g$, $\mathrm{tr}_{F/Q}(\lambda z)=\sum_{1\leq i\leq g}\lambda^{(i)}z_i$, and $\mathfrak{d}$ is the different of $F/\boldsymbol{Q}$. By the assumption (b), $\mathfrak{d}=(\delta)$ with $\delta\gg 0$. As in [5] [6] [7] [11], we put $[f]=E_{2,1}^k(*,f)$ if $\Phi f=0$ and $[f]=E_{2,0}^k(*,\Phi f)$ if $\Phi f\neq 0$. Let $[f](Z)=\sum_{T\geqslant 0}a(T,[f])\boldsymbol{e}(\sigma(\mathrm{tr}_{F/Q}(\delta^{-1}TZ)))$ be the Fourier expansion of $[f]$, where T runs over all symmetric totally positive semi-definite semi-integral (i.e. $T=(t_{ij})$, $2t_{ij}\in\mathcal{O}_F$, $t_{ii}\in\mathcal{O}_F$ for $1\leqq i,j\leqq 2$) matrices of size 2, and σ is the trace of matrices. To obtain a formula for $a(T,[f])$, it is sufficient to consider the case $T\gg 0$. We denote by $d(F)$ the discriminant of F.

Theorem 1. *For $T\gg 0$ such that $|2T|$ is square-free* (i.e., *$|2T|$ is not divisible by the square of any proper ideal in $\mathcal{O}_F$*), *we have*:

$$a(T,[f])=\frac{1}{2}(-1)^{kg/2}\left(2(2\pi)^{k-1}\frac{(k-1)!}{(2k-2)!}\right)^g$$

$$\cdot \mathrm{N}(|2T|)^{k-(3/2)} d(F)^{1-k} \frac{L_F(k-1,\chi)D(k-1,f,\vartheta_T)}{L_2(2k-2,f)}.$$

Here $g=(F:\boldsymbol{Q})$, χ denotes the Hecke character attached to the quadratic extension $F(\sqrt{-|2T|})/F$, $L_F(s,\chi)$ the Hecke L-function, and $\vartheta_T(z) = \sum_{\varepsilon:E+/E^2} \sum_{(\lambda,\mu)\in\mathcal{O}_F\times\mathcal{O}_F} e\left(\mathrm{tr}_{F/Q}\left(z\varepsilon\delta^{-1}(\lambda\mu)T\binom{\lambda}{\mu}\right)\right)$. We take complex numbers $\alpha_\mathfrak{p}$, $\beta_\mathfrak{p}$ such that $\sum_\mathfrak{a} a(\mathfrak{a})\mathrm{N}(\mathfrak{a})^{-s} = \prod_\mathfrak{p}(1-\alpha_\mathfrak{p}\mathrm{N}(\mathfrak{p})^{-s})^{-1}(1-\beta_\mathfrak{p}\mathrm{N}(\mathfrak{p})^{-s})^{-1}$, where $\mathfrak{a}$ runs over all integral ideals of F and $\mathfrak{p}$ runs over all prime ideals of F, and put $L_2(s,f)=\prod_\mathfrak{p}(1-\alpha_\mathfrak{p}^2\mathrm{N}(\mathfrak{p})^{-s})^{-1}(1-\alpha_\mathfrak{p}\beta_\mathfrak{p}\mathrm{N}(\mathfrak{p})^{-s})^{-1}(1-\beta_\mathfrak{p}^2\mathrm{N}(\mathfrak{p})^{-s})^{-1}$. Writing $\vartheta_T(z)=\sum_{0\ll\lambda\in\mathfrak{d}^{-1}} b((\lambda)\mathfrak{d})e(\mathrm{tr}_{F/Q}(\lambda z))$, we put $D(s,f,\vartheta_T)=\sum_\mathfrak{a} a(\mathfrak{a})b(\mathfrak{a})\mathrm{N}(\mathfrak{a})^{-s}$. Each L-function is considered as a meromorphic function on $\boldsymbol{C}$ by the analytic continuation. If $\Phi f\neq 0$, then $D(s,f,\vartheta_T)$ and $L_2(2s,f)$ have zeros of the same order at $s=k-1$, and we understand that $D(k-1,f,\vartheta_T)/L_2(2k-2,f)=\lim_{s\to k-1} D(s,f,\vartheta_T)/L_2(2s,f)$.

Remark 1. By the condition (a) in § 1, kg is an even integer.

Remark 2. If we drop the condition (c) on F, then we have the following: $a(T,[f])$ has a similar expression as in Theorem 1, with ϑ_T replaced by some h, a Hilbert modular form of weight 1 and type $(|2T|,\chi)$ whose Fourier coefficients lie in the totally real number field generated over $\boldsymbol{Q}$ by the eigen values of all Hecke operators on f.

Suppose $\Phi f\neq 0$, i.e. $f(z)=G_k(z)=((k-1)!(2\pi\sqrt{-1})^{-k})^g d(F)^{k-(1/2)}\zeta_F(k) + \sum_{0\ll\nu\in\mathfrak{d}^{-1}} \sigma_{k-1}(\nu)e(\mathrm{tr}_{F/Q}(\nu z))$ where $\zeta_F(s)$ is the Dedekind zeta function of F and $\sigma_{k-1}(\nu)=\sum_{\mathfrak{a}|(\nu)\mathfrak{d}} \mathrm{N}(\mathfrak{a})^{k-1}$ ($\mathfrak{a}$ running over all integral ideals of F that divide $(\nu)\mathfrak{d}$). In this case we obtain the following generalization of the above formula by a method similar to that of Maaß [9]:

Theorem 2. *Let F, χ, G_k be as above. Suppose $T\gg 0$ is primitive* $\left(\text{i.e. } T=\begin{pmatrix} t_1 & t/2 \\ t/2 & t_2 \end{pmatrix}\right.$ *with* $\left.t_1, t_2, t\in\mathcal{O}_F,\ (t_1,t_2,t)=1\right)$. *Then we have:*

$$a(T,[G_k])=(-1)^{kg/2}\left(\frac{(k-1)!}{(2k-2)!}2(2\pi)^{k-1}\right)^g \mathrm{N}(|2T|)^{k-(3/2)} d(F)^{1-k} \frac{L_F(k-1,\chi)}{\zeta_F(2k-2)}$$
$$\cdot \prod_{\mathfrak{p}|2|2T|} \left\{(1-\chi(\mathfrak{p})\mathrm{N}(\mathfrak{p})^{1-k}) \sum_{0\leq\mu\leq j(\mathfrak{p})} \mathrm{N}(\mathfrak{p})^{\mu(3-2k)} + \gamma(b/2)\mathrm{N}(\mathfrak{p})^{b(3-2k)/2}\right\}.$$

Here b is the maximal integer such that $\mathfrak{p}^b \| |2T|$; $j(\mathfrak{p})=[(b-1)/2]$ if $\mathfrak{p}\nmid 2$, $j(\mathfrak{p})=[(b-2)/2]$ if $\mathfrak{p}|2$; and $\gamma(x)=1$ or 0 according as x is an integer or not.

Remark 3. The above formula coincides with the formula of Maaß [9] if $F=\boldsymbol{Q}$.

§ 3. An application. We apply Theorem 1 to investigate the value $L_2(2k-2,f)$. This is an application of type (I) stated in [7]. Let F be a totally really real number field of degree $g=(F:\boldsymbol{Q})$ with the class number one in the narrow sense. Let $k>0$ be an integer satisfying (a) in § 1 and $f\in S_k(\Gamma_F^{(1)})$ be a normalized eigen Hilbert cusp form.

Then, as in Kurokawa [6], Harris [3], and Garrett [2], we have $a(T, [f]) \in \bar{\boldsymbol{Q}}$ for all $T \gg 0$, where $\bar{\boldsymbol{Q}}$ denotes the algebraic closure of $\boldsymbol{Q}$ in $\boldsymbol{C}$. (The author received Garrett's preprint [2] in October 1981 after the first draft of this paper was prepared.) Moreover we know by Theorem 1 (after a short argument) that for any $T \gg 0$ with $|2T|$ square-free integer in $\mathcal{O}_F$ there exists some $T_1 \gg 0$ such that $|2T_1| = |2T|$ and that $a(T_1, [f]) \neq 0$. Hence, by Theorem 1 and Remark 2 combined with a result of Shimura [12] on $D(k-1, f, h)$, we have the following:

Theorem 3. *Let F, g, k, and f be as above. Then:*

$$L_2(2k-2, f)/\pi^{(3k-2)g}\langle f, f\rangle \in \bar{\boldsymbol{Q}}.$$

Here, $\langle f, f\rangle$ denotes the normalized Petersson norm, i.e. $\langle f, f\rangle = \mathrm{vol}(\mathfrak{F})^{-1}\int_{\mathfrak{F}} |f(z)|^2\, \mathrm{Im}(z)^k d\mu(z)$, where $\mathfrak{F}$ is a fundamental domain of $\Gamma_F^{(1)}\backslash\mathfrak{H}_1^g$, $\mathrm{Im}(z) = \prod_{1\leq i\leq g} y_i$ and $d\mu(z) = \prod_{1\leq i\leq g} y_i^{-2} dx_i dy_i$ if $z = (z_1, \cdots, z_g)$, $z_i = x_i + \sqrt{-1}\, y_i$.

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