63. On Boundedness of Circular Domains*)

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Introduction. The main purpose of this note is to prove the following assertions:

(I) *The classification problem for generalized Siegel domains in $C\times C^m$ in the sense of Kaup, Matsushima and Ochiai* [3] *can be completely reduced to that for bounded circular domains in C^N, where $N < m+1$* (Theorem 1);

(II) *Let D be a starlike circular domain in C^n. Then D is Kobayashi hyperbolic if and only if it is a bounded domain in C^n* (Theorem 2).

(I) is a supplement to our previous papers [5], [7]. (II) gives a partial affirmative answer to the following fundamental problem in the theory of hyperbolic manifolds: *If D is a domain in C^n and it is hyperbolic in the sense of Kobayashi* [4], *then is it true that D is holomorphically equivalent to a bounded domain in C^n?* Recently, Barth [1] obtained an affirmative answer to this problem in the case where D is a geometrically convex domain.

The author would like to express his thanks to his colleague K. Azukawa who presents him an interesting example in § 3.

1. The structure of generalized Siegel domains in $C\times C^m$. Let $\mathcal{D}$ be a generalized Siegel domain in $C\times C^m$ with exponent c. Let $\mathrm{Aut}(\mathcal{D})$ be the group of all biholomorphic transformations of $\mathcal{D}$ onto itself and $\mathfrak{g}(\mathcal{D})$ the Lie algebra of all complete holomorphic vector fields on $\mathcal{D}$. Then it is known [3] that $\mathfrak{g}(\mathcal{D})$ is identified with the Lie algebra of $\mathrm{Aut}(\mathcal{D})$ and it has a canonical graduation

$$\mathfrak{g}(\mathcal{D})=\mathfrak{g}_{-1}+\mathfrak{g}_{-1/2}+\mathfrak{g}_0+\mathfrak{g}_{1/2}+\mathfrak{g}_1, \qquad [\mathfrak{g}_\lambda, \mathfrak{g}_\mu]\subset\mathfrak{g}_{\lambda+\mu}$$

and

$$\dim_R \mathfrak{g}_{-1/2}=2k$$

for some k, $0\leq k\leq m$.

Theorem 1. *Let $\mathcal{D}$ be a generalized Siegel domain in $C\times C^m$ with exponent c and $\dim_R \mathfrak{g}_{-1/2}=2k$. Then we have the following*

(1) *If $c=1/2$, $\mathcal{D}$ can be transformed by a non-singular linear mapping to a canonical form*

$$D=\Big\{(z, w_1, \cdots, w_m)\in C\times C^m\,;\ \mathrm{Im}\, z-\sum_{\alpha=1}^{k}|w_\alpha|^2>0,$$

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$$\left(\frac{w_{k+1}}{(\operatorname{Im} z-\sum_{\alpha=1}^{k}|w_\alpha|^2)^{1/2}}, \cdots, \frac{w_m}{(\operatorname{Im} z-\sum_{\alpha=1}^{k}|w_\alpha|^2)^{1/2}}\right) \in D_{\sqrt{-1}}\Big\},$$

where

$$D_{\sqrt{-1}}=\{(w_{k+1}, \cdots, w_m) \in \boldsymbol{C}^{m-k};\ (\sqrt{-1}, 0, \cdots, 0, w_{k+1}, \cdots, w_m) \in D\}$$

is a bounded circular domain in $\boldsymbol{C}^{m-k}$ *containing the origin;*

(2) *If* $c \neq 1/2$, *then* $\dim_{\boldsymbol{R}} \mathfrak{g}_{-1/2}=\dim_{\boldsymbol{R}} \mathfrak{g}_{1/2}=0$ *and*

$$\mathcal{D}=\{(z, w) \in \boldsymbol{C}\times\boldsymbol{C}^m;\ \operatorname{Im} z>0,\ w/(\operatorname{Im} z)^c \in \mathcal{D}_{\sqrt{-1}}\},$$

where $\mathcal{D}_{\sqrt{-1}}=\{w \in \boldsymbol{C}^m;\ (\sqrt{-1}, w) \in \mathcal{D}\}$ *is a bounded circular domain in* $\boldsymbol{C}^m$ *containing the origin.*

Proof. The only thing which has to be proven now is that the circular domains $D_{\sqrt{-1}}$ and $\mathcal{D}_{\sqrt{-1}}$ are bounded. Indeed, in [5], [7] we have already shown the other assertions in the theorem. Now, in order to prove the boundedness of these circular domains, we may assume that $D=\mathcal{D}$ in the theorem. Under this assumption we consider a mapping $\varphi:\{z \in \boldsymbol{C};\ \operatorname{Im} z>0\}\times\boldsymbol{C}^m \to \boldsymbol{C}^{m+1}$ defined by

$$z_1=(z-\sqrt{-1})\cdot(z+\sqrt{-1})^{-1}, \qquad z_k=\frac{4^c\cdot w_{k-1}}{(z+\sqrt{-1})^{2c}} \tag{1.1}$$

for $k=2, 3, \cdots, m+1$. Then φ is injective and holomorphic on $\mathcal{D}$. Hence it defines a biholomorphic isomorphism of $\mathcal{D}$ onto $\mathcal{B}=\varphi(\mathcal{D})$ in $\boldsymbol{C}^{m+1}$. Here we assert that

(1.2) *$\mathcal{B}$ is a bounded circular domain in $\boldsymbol{C}^{m+1}$ containing the origin.*

Indeed, we can show with exactly the same arguments as in [6, Lemma 1] that $\mathcal{B}$ is a circular domain in $\boldsymbol{C}^{m+1}$ with center o which is holomorphically equivalent to a bounded domain in $\boldsymbol{C}^{m+1}$. Thus the assertion (1.2) is an immediate consequence of [2, Théorème V]. Now, we put

$$\mathcal{B}_o=\{z \in \boldsymbol{C}^m;\ (0, z) \in \mathcal{B}\} \quad\text{and}\quad \mathcal{D}_o=\{w \in \boldsymbol{C}^m;\ (\sqrt{-1}, w) \in \mathcal{D}\}.$$

Then $\mathcal{B}_o$ is a bounded circular domain in $\boldsymbol{C}^m$ by (1.2) and $\mathcal{D}_o$ is a circular domain in $\boldsymbol{C}^m$. On the other hand, it follows from (1.1) that the restriction $\varphi|_{\{\sqrt{-1}\}\times\boldsymbol{C}^m}: \{\sqrt{-1}\}\times\boldsymbol{C}^m \to \boldsymbol{C}^{m+1}$ is given by $(\sqrt{-1}, w) \mapsto (0, w/(\sqrt{-1})^{2c})$, from which $\mathcal{D}_o=\mathcal{B}_o$. Therefore $\mathcal{D}_o$ is also bounded. Since $\mathcal{D}_{\sqrt{-1}}=\mathcal{D}_o$ and $D_{\sqrt{-1}}\subset\mathcal{D}_o$ via the natural identification, we finally conclude that $D_{\sqrt{-1}}$ and $\mathcal{D}_{\sqrt{-1}}$ are bounded, completing the proof.

2. Circular domains and Kobayashi hyperbolicity. Let M be a complex analytic space and d_M the Kobayashi pseudodistance of M.

Theorem 2. *Let D be a starlike circular domain in $\boldsymbol{C}^n$. Then D is hyperbolic if and only if it is a bounded domain in $\boldsymbol{C}^n$.*

Proof. We may assume that D is a circular domain with center o, the origin of $\boldsymbol{C}^n$. Since it is well-known [4] that a bounded domain is hyperbolic, we have only to prove the converse.

Suppose that D is unbounded. Then we may obtain a sequence $\{z_k\}_{k=1}^{\infty}$ of points $z_k \in D$ such that $|z_k|>1$ and $|z_k|\to\infty$ as $k\to\infty$, where $|\cdot|$

denotes the Euclidean norm on C^n. For this sequence we define a mapping $f_k: \Delta \to C^n$ by

$$f_k(t) = t \cdot z_k, \quad t \in \Delta \quad \text{for } k=1, 2, 3, \cdots,$$

where $\Delta = \{t \in C; |t| < 1\}$ is the unit disk in C. Since D is a starlike circular domain with center o and $z_k \in D$, we see that every f_k is a holomorphic mapping of Δ into D. Now, taking $\varepsilon > 0$ in such a way that $0 < \varepsilon < 1$ and the ε-sphere $S(\varepsilon) = \{z \in C^n; |z| = \varepsilon\}$ is contained in D, we consider the sequences of points

$$a_k = (\varepsilon/|z_k|) \cdot z_k \quad \text{and} \quad b_k = \varepsilon/|z_k| \quad \text{for } k=1, 2, 3, \cdots.$$

Since $\varepsilon/|z_k| < 1$ for every k, we have

$$\begin{cases} a_k \in S(\varepsilon), \quad b_k \in \Delta, \quad f_k(b_k) = a_k \\ \text{for } k=1, 2, 3, \cdots, \quad \text{and} \quad \lim_{k \to \infty} b_k = 0. \end{cases}$$

By the distance decreasing property of holomorphic mappings with respect to the Kobayashi pseudodistances, it then follows that

$$d_D(a_k, o) = d_D(f_k(b_k), f_k(o)) \leqq d_\Delta(b_k, o) \longrightarrow 0 \tag{2.1}$$

as $k \to \infty$. On the other hand, passing to a subsequence if necessary, we may assume that $\{a_k\}_{k=1}^\infty$ converges to a point a of $S(\varepsilon) \subset D$. By the continuity of d_D and (2.1) we conclude that $d_D(a, o) = 0$. Obviously this says that D is not hyperbolic. Q.E.D.

Since a pseudoconvex circular domain is starlike, the following corollary is an immediate consequence of our theorem.

Corollary 1. *Let D be a pseudoconvex circular domain in C^n. Then D is hyperbolic if and only if it is bounded.*

Corollary 2. *Let D be a homogeneous circular domain in C^n. Then the following conditions are mutually equivalent:*

(1) *D is hyperbolic;*

(2) *D admits an Aut(D)-invariant Hermitian metric;*

(3) *D is a bounded symmetric domain.*

Proof. Recall that a homogeneous hyperbolic domain C^n is complete hyperbolic, and hence it is pseudoconvex by [4, p. 77, Theorem 3.4]. Therefore, the equivalence of (1) and (3) follows from Corollary 1 and the fact that any homogeneous bounded circular domain is symmetric. Next, assume the condition (2). Then, from [7, Lemma 1.2] we see that orbit $D = \mathrm{Aut}_o(D) \cdot p$ passing through the center p of D is a Hermitian symmetric space of non-compact type, where $\mathrm{Aut}_o(D)$ denotes the identity component of Aut(D). Then, it follows from [2, Théorème V] that D is also bounded, proving (3). Finally, the implication (3)$\to$(2) is well-known. Q.E.D.

3. Example. Modifying the results of Sadullaev [8] and Barth [1], K. Azukawa has obtained the following example, from which we see that there exists a non-hyperbolic pseudoconvex circular domain in C^n containing no complex lines. This may be interesting when it is compared with the result of Barth [1].

Example. We put for $z \in C$

$$v(z)=\max.\left\{\log|z|,\ \sum_{k=2}^{\infty}\frac{1}{k^2}\cdot\log\left|z-\frac{1}{k}\right|\right\}.$$

Then $v(z)$ is a real-valued subharmonic function on C. Putting

$$R(z^1, z^2)=\begin{cases}(\exp(-v(z^1/z^2))\cdot\sqrt{1+|z^1/z^2|^2}, & z^2\neq 0\\ 1, & z^2=0,\end{cases}$$

we now define a domain D in C^2 by

$$D=\{z\in C^2;\ |z|<R(z)\}.$$

Then it can be seen that D is an unbounded pseudoconvex circular domain (and hence it is not hyperbolic by Corollary 1) which contains no complex lines.

References

[1] T. J. Barth: Convex domains and Kobayashi hyperbolicity. Proc. Amer. Math. Soc., **79**, 556–558 (1980).

[2] H. Cartan: Les fonctions de deux variables complexes et le problème de la représentation analytique. J. Math. Pures Appl., **10**, 1–114 (1931).

[3] W. Kaup, Y. Matsushima, and T. Ochiai: On the automorphisms and equivalences of generalized Siegel domains. Amer. J. Math., **92**, 475–498 (1970).

[4] S. Kobayashi: Hyperbolic Manifolds and Holomorphic Mappings. Marcel Dekker, New York (1970).

[5] A. Kodama: On generalized Siegel domains. Osaka J. Math., **14**, 227–252 (1977).

[6] ——: On the equivalence problem for a certain class of generalized Siegel domains. II. ibid., **16**, 113–142 (1979).

[7] ——: ditto. III. ibid., **18**, 481–499 (1981).

[8] A. Sadullaev: Schwarz lemma for circular domains and its applications. Math. Notes of Aca. Sci. USSR, **27**, 120–125 (1980).