21. *On the Trotter Product Formula*

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(Communicated by Kôsaku YOSIDA, M. J. A., Feb. 12, 1981)

Introduction. Kato [5] (cf. Kato-Masuda [8]) proved the Trotter product formula $s\text{-}\lim_{n\to\infty}[e^{-tA/n}e^{-tB/n}]^n = e^{-t(A\dot{+}B)}P$ for the form sum $A\dot{+}B$ of self-adjoint operators A and B which are bounded from below in a Hilbert space $\mathcal{H}$. Here P is the orthogonal projection of $\mathcal{H}$ onto the closure of $\mathcal{D}(|A|^{1/2})\cap\mathcal{D}(|B|^{1/2})$. The purpose of this paper is to extend this result to prove a product formula for the form sum of self-adjoint operators which are not necessarily bounded from below. The product formula obtained involves a "truncation" procedure.

1. Notations and results. First we consider the case of two operators. Let A and B be self-adjoint operators in a Hilbert space $\mathcal{H}$ with spectral families $\{E_A(\lambda)\}$ and $\{E_B(\lambda)\}$, respectively. Let A_+ and A_- be the positive and negative parts of A, i.e. $A_+ = AE_A([0,\infty))\geqslant 0$, $A_- = -AE_A((-\infty,0))\geqslant 0$, and $A = A_+ - A_-$. Define B_+ and B_- similarly for B.

Assume that $\mathcal{D}(A_+^{1/2})\subset\mathcal{D}(B_-^{1/2})$ and $\mathcal{D}(B_+^{1/2})\subset\mathcal{D}(A_-^{1/2})$, and that there exist constants $\alpha\geqslant 0$ and $0\leqslant\beta<1$ such that

$$\begin{aligned} \|A_-^{1/2}u\|^2 &\leqslant \alpha\|u\|^2+\beta\|B_+^{1/2}u\|^2, \qquad u\in\mathcal{D}(B_+^{1/2}),\\ \|B_-^{1/2}u\|^2 &\leqslant \alpha\|u\|^2+\beta\|A_+^{1/2}u\|^2, \qquad u\in\mathcal{D}(A_+^{1/2}). \end{aligned} \tag{1}$$

Set $\mathcal{D}=\mathcal{D}(A_+^{1/2})\cap\mathcal{D}(B_+^{1/2})$, and let P be the orthogonal projection of $\mathcal{H}$ onto the closure $\overline{\mathcal{D}}$ of $\mathcal{D}$. Then the quadratic form

$$u\mapsto\|A_+^{1/2}u\|^2+\|B_+^{1/2}u\|^2-\|A_-^{1/2}u\|^2-\|B_-^{1/2}u\|^2, \qquad u\in\mathcal{D}, \tag{2}$$

is bounded from below and closed. The *form sum* of A and B is defined as the self-adjoint operator in the Hilbert space $\overline{\mathcal{D}}$ associated with (2) and denoted by $A\dot{+}B$.

For each $0<\tau\leqslant\infty$, $\mathcal{F}(\tau)$ is the class of bounded real-valued functions $h(t,\lambda)$ on $[0,\tau)\times\boldsymbol{R}$ satisfying the following conditions:

(i) for each fixed λ, $h(t,\lambda)$ is continuous in t at $t=0$ with
$$h(0,\lambda)=1, \qquad (\partial/\partial t)h(0,\lambda)=-\lambda;$$

(ii) for each fixed t, $h(t,\lambda)$ is Borel measurable in λ with
$$1\leqslant h(t,\lambda) \text{ for } \lambda<0,\ h(t,0)=1 \text{ and } 0\leqslant h(t,\lambda)\leqslant 1 \text{ for } \lambda>0;$$

(iii) there is a constant M such that $|1-h(t,\lambda)|\leqslant Mt|\lambda|$, $0\leqslant t<\tau$, $\lambda\in\boldsymbol{R}$.

The main result is the following product formula.

Theorem 1. *Let $f(t,\lambda)$ and $g(t,\lambda)$ be in $\mathcal{F}(\tau)$ for some $0<\tau\leqslant\infty$, and assume that there exists a constant $z>1$ such that*

$$\beta \sup_{\lambda<0}(t\lambda)^{-1}(1-f(t,\lambda)^{2z}) \leqslant \inf_{\lambda>0}(t\lambda)^{-1}(g(t,\lambda)^{-2}-1),\ 0<t<\tau,$$
$$\beta \sup_{\lambda<0}(t\lambda)^{-1}(1-g(t,\lambda)^{2z}) \leqslant \inf_{\lambda>0}(t\lambda)^{-1}(f(t,\lambda)^{-2}-1),\ 0<t<\tau. \quad (3)$$

Then

$$[f(t/n, A)g(t/n, B)]^n \xrightarrow[s]{} e^{-t(A\dot{+}B)}P, \quad n\to\infty, \quad t>0. \quad (4)$$

The convergence is uniform in $t\in[0, T]$ *for every* $T>0$ *when applied to* $u\in\overline{\mathcal{D}}$, *and in* $t\in[T', T]$ *for every* $0<T'<T$ *when applied to* $u\perp\mathcal{D}$.

Examples. For each $0<\tau\leqslant\infty$, $\mathcal{F}(\tau)$ includes the following functions obtained by truncating the functions $e^{-t\lambda}$ and $(1+t\lambda/k)^{-k}$, $k=1, 2, \cdots$, where $\lambda<-\delta/t$:

$$e^{\delta}\chi_{(-\infty,-\delta)}(t\lambda)+e^{-t\lambda}\chi_{[-\delta,\infty)}(t\lambda), \quad (5)$$
$$e^{-ta}\chi_{(-\infty,-\delta)}(t\lambda)+e^{-t\lambda}\chi_{[-\delta,\infty)}(t\lambda), \quad (6)$$
$$(1-\delta/k)^{-k}\chi_{(-\infty,-\delta)}(t\lambda)+(1+t\lambda/k)^{-k}\chi_{[-\delta,\infty)}(t\lambda), \quad (7)$$
$$(1+ta/k)^{-k}\chi_{(-\infty,-\delta)}(t\lambda)+(1+t\lambda/k)^{-k}\chi_{[-\delta,\infty)}(t\lambda). \quad (8)$$

Here δ and a are arbitrary constants with $0<\delta<k$ and $-\delta/\tau\leqslant a\leqslant 0$ where $-\delta/\tau=0$ if $\tau=\infty$, and $\chi_K(x)$ denotes the characteristic function of $K\subset\boldsymbol{R}$. Moreover if δ is so chosen that $\beta((1-\delta/k)^{-2k}-1)<2\delta$, then each pair of the functions (5)–(8) satisfies the condition (3) with $z=-\log(1+2\delta/\beta)/2k\log(1-\delta/k)>1$. Thus Theorem 1 is applicable.

Remark 1. If A (resp. B) is bounded from below, $f(t,\lambda)$ (resp. $g(t,\lambda)$) needs only to satisfy the conditions (i)–(iii) of $\mathcal{F}(\tau)$ as a bounded real-valued function defined on $[0,\tau)\times[\inf\sigma(A),\infty)$ (resp. $[0,\tau)\times[\inf\sigma(B),\infty)$). Here $\sigma(A)$ and $\sigma(B)$ denote the spectra of A and B. Thus Theorem 1 includes Kato's result [5] for both A and B nonnegative; the condition (3) is trivially satisfied with $\beta=0$.

Remark 2. The condition $\beta<1$ in (1) is necessary for $z>1$. In fact, we see by the condition (i) of $\mathcal{F}(\tau)$ that $\beta z\leqslant 1$, letting $t\downarrow 0$ in (3).

Remark 3. If $f(t,\lambda)$ and $g(t,\lambda)$ are in $\mathcal{F}(\infty)$ and satisfy (3), it will be seen in the proof of Theorem 1 that the approximant operators in (4) are uniformly quasi-bounded, i.e. $\|[f(t/n,A)g(t/n,B)]^n\|\leqslant Ce^{\gamma t}$, $t>0$, $n=1,2,\cdots$, with some constants C and γ. However, for instance, $[e^{-tA/n}e^{-tB/n}]^n$ may not be uniformly quasi-bounded as is seen in the next example. The essence of the theorem is that a product formula holds if those truncated functions (5) and (6) are used instead of $e^{-t\lambda}$. In this connection we also refer to Ichinose [3].

Example. Let $\mathcal{H}=L^2(\boldsymbol{R}^l)$. Let $V(x)$ be a real-valued measurable function on $\boldsymbol{R}^l$, and let Δ be the l-dimensional Laplacian. If $\|[e^{-tV/n}e^{t\Delta/n}]^n\|\leqslant Ce^{\gamma t}$, $t>0$, $n=1,2,\cdots$, then $-\gamma\leqslant V(x)$ a.e. on $\boldsymbol{R}^l$. In fact, we need only to show that for every $R>0$ and $\varepsilon>0$, the measure $m(K(R,\varepsilon))$ of $K(R,\varepsilon)=\{x\in\boldsymbol{R}^l;\ V(x)<-\gamma-\varepsilon,\ |x|\leqslant R\}$ is zero. Note that

$$\begin{aligned}
&[e^{-tV(x)}e^{t\Delta}]^n\chi_{K(R,\varepsilon)}(x)\\
&\quad\geqslant[e^{-tV(x)}e^{t\Delta}]^{n-1}e^{(\gamma+\varepsilon)t-R^2/t}(4\pi t)^{-l/2}m(K(R,\varepsilon))\chi_{K(R,\varepsilon)}(x)\\
&\quad\geqslant e^{n(\gamma+\varepsilon)t-nR^2/t}(4\pi t)^{-nl/2}m(K(R,\varepsilon))^n\chi_{K(R,\varepsilon)}(x).
\end{aligned}$$

Thus if $m(K(R,\varepsilon))\neq 0$, we have $C^{1/n}e^{-t\varepsilon+R^2/t}(4\pi t)^{l/2}\geqslant m(K(R,\varepsilon))$, $t>0$, by assumption. But it follows by letting $t\to\infty$ that $m(K(R,\varepsilon))=0$. This is a contradiction.

Next consider the case of several operators. For each $j=1,\cdots,m$, let A_j be a self-adjoint operator in $\mathscr{H}$ with spectral family $\{E_j(\lambda)\}$. Define the positive and negative parts $A_{j,+}$ and $A_{j,-}$ of A_j as before.

Assume that, for each $j=1,\cdots,m$, $\mathscr{D}(A_{j,+}^{1/2})\subset\mathscr{D}(A_{j+1,-}^{1/2})$, and that there exist constants $\alpha\geqslant 0$ and $0\leqslant\beta<1$ such that

$$\|A_{j+1,-}^{1/2}u\|^2\leqslant\alpha\|u\|^2+\beta\|A_{j,+}^{1/2}u\|^2,\qquad u\in\mathscr{D}(A_{j,+}^{1/2}),\tag{9}$$

where $A_{m+1}=A_1$. Set $\mathscr{D}=\bigcap_{j=1}^{m}\mathscr{D}(A_{j,+}^{1/2})$. Then the quadratic form

$$u\mapsto\sum_{j=1}^{m}\|A_{j,+}^{1/2}u\|^2-\sum_{j=1}^{m}\|A_{j,-}^{1/2}u\|^2,\qquad u\in\mathscr{D},\tag{10}$$

is bounded from below and closed. The *form sum* $A_1\dot{+}\cdots\dot{+}A_m$ of the A_j, $j=1,\cdots,m$, is defined as the self-adjoint operator in the Hilbert space $\bar{\mathscr{D}}$ associated with (10).

We avoid inessential complication and content ourselves with a rather small class of functions which is included in $\mathscr{F}(\tau)$, and which contains the functions (5)–(8).

Theorem 2. *Let $0<\tau\leqslant\infty$. For each $j=1,\cdots,m$, let $f_j(t,\lambda)$ be a bounded nonnegative function defined on $[0,\tau)\times\boldsymbol{R}$ of the form*

$$f_j(t,\lambda)=k_j(t)\chi_{(-\infty,-\delta)}(t\lambda)+f_j(t\lambda)\chi_{[-\delta,\infty)}(t\lambda),\qquad\delta>0,$$

where (i) *each $f_j(s)$ is a bounded nonnegative and Borel measurable function on $[-\delta,\infty)$ satisfying*

$$[1-(\zeta s)^{3/2}]/[1+\zeta s+(\zeta s)^2]\leqslant f_j(s)^\zeta\leqslant[1+(\zeta s)^{3/2}]/[1+\zeta s+(\zeta s)^2],\tag{11}$$

for $s\geqslant 0$ with $\zeta=1$, and for $-\delta\leqslant s<0$ with all ζ in some common nonempty open interval $I\subset(-\infty,0)$, and (ii) *each $k_j(t)$ is a function on $[0,\tau)$ satisfying $1\leqslant k_j(t)\leqslant f_j(-\delta)$. Assume that there exists a constant $z>1$ such that,*

$$\beta\sup_{-\delta\leqslant s<0}s^{-1}(1-f_{j+1}(s)^{2z})\leqslant\inf_{s>0}s^{-1}(f_j(s)^{-2}-1),\quad j=1,\cdots,m,\tag{12}$$

where $f_{m+1}(s)=f_1(s)$. Then for $u\in\bar{\mathscr{D}}$,

$$[f_m(t/n,A_m)\cdots f_1(t/n,A_1)]^n u\to\exp[-t(A_1\dot{+}\cdots\dot{+}A_m)]u,\quad n\to\infty,\ t\geqslant 0.\tag{13}$$

The convergence is uniform in $t\in[0,T]$ for every $T>0$.

Theorem 2 is somewhat weak compared with Theorem 1. The convergence in (13) for $u\perp\mathscr{D}$ seems to remain unknown (cf. [8]).

2. Proof of theorems. Proof of Theorem 1. We shall use the method of Kato [4, 5] and Simon [5, Addendum] with Vitali's theorem.

For $K\subset\boldsymbol{R}$, let $\mathscr{B}(K,\mathscr{H})$ be the Banach space of all bounded $\mathscr{H}$-valued functions on K. For $\zeta\in\boldsymbol{C}$, $0\leqslant t<\tau$ and $\lambda\in\boldsymbol{R}$ put

$$\begin{aligned}f(\zeta,t,\lambda)&=f(t,\lambda)^\zeta\chi_{(-\infty,0)}(t\lambda)+f(t,\lambda)\chi_{[0,\infty)}(t\lambda),\\ g(\zeta,t,\lambda)&=g(t,\lambda)^\zeta\chi_{(-\infty,0)}(t\lambda)+g(t,\lambda)\chi_{[0,\infty)}(t\lambda).\end{aligned}\tag{14}$$

Put

$$U(\zeta,t)=f(\zeta,t,A)g(\zeta,t,B).$$

The proof is divided into five steps. Let $0<T'<T$.

I. It is easy to see that if $n>T/\tau$ and $u\in\mathcal{H}$ then $U(\zeta, t/n)^n u$ is holomorphic in ζ as a $\mathcal{B}([0, T], \mathcal{H})$-valued function.

II. There exist constants C and $\gamma\geqslant 0$ such that, for each n with $n>T/\tau$ and for each ζ with $\operatorname{Re}\zeta<z$, $\|U(\zeta, t/n)^n\|\leqslant Ce^{\gamma t}$, $0\leqslant t\leqslant T$.

To show this, first note $f(\zeta, t, A)=f(\zeta, t, A_+)f(\zeta, t, -A_-)$ with

$$f(\zeta, t, A_+)=E_A((-\infty, 0))+\int_R f(t, \lambda)\chi_{[0,\infty)}(\lambda)dE_A(\lambda),$$

$$f(\zeta, t, -A_-)=\int_R f(t, \lambda)^\zeta\chi_{(-\infty,0)}(\lambda)dE_A(\lambda)+E_A([0, \infty)),$$

and similarly for $g(\zeta, t, B)$. For $0<t<\tau$, put

$$M(f, t)=\sup_{\lambda<0}(t\lambda)^{-1}(1-f(t, \lambda)^{2z}),\quad M(g, t)=\sup_{\lambda<0}(t\lambda)^{-1}(1-g(t, \lambda)^{2z}).$$

By the condition (iii) of $\mathcal{F}(\tau)$ and (3), both $M(f, t)$ and $M(g, t)$ are bounded by some constant M and $\beta M(f, t)t\lambda g(t, \lambda)^2\leqslant 1-g(t, \lambda)^2$, $0<t<\tau$, $\lambda\geqslant 0$. Then for $u\in\mathcal{H}$ we have in view of (1)

$$\begin{aligned}
&\|f(\zeta, t, -A_-)g(\zeta, t, B_+)u\|^2\\
&\leqslant\int_R [f(t, \lambda)^{2z}\chi_{(-\infty,0)}(t\lambda)+\chi_{[0,\infty)}(t\lambda)]d\|E_A(\lambda)g(\zeta, t, B_+)u\|^2\\
&\leqslant\int_R [M(f, t)t|\lambda|\chi_{(-\infty,0)}(t\lambda)+1]d\|E_A(\lambda)g(\zeta, t, B_+)u\|^2\\
&=M(f, t)t\|A_-^{1/2}g(\zeta, t, B_+)u\|^2+\|g(\zeta, t, B_+)u\|^2\\
&\leqslant\beta M(f, t)t\|B_+^{1/2}g(\zeta, t, B_+)u\|^2+(1+\alpha M(f, t)t)\|g(\zeta, t, B_+)u\|^2\\
&=\int_R [(\beta M(f, t)t\lambda+1+\alpha M(f, t)t)g(t, \lambda)^2\chi_{[0,\infty)}(t\lambda)\\
&\quad+(1+\alpha M(f, t)t)\chi_{(-\infty,0)}(t\lambda)]d\|E_B(\lambda)u\|^2\\
&\leqslant(1+\alpha M(f, t)t)\|u\|^2\leqslant(1+\alpha Mt)\|u\|^2\leqslant e^{\alpha Mt}\|u\|^2.
\end{aligned}$$

Thus $\|f(\zeta, t, -A_-)g(\zeta, t, B_+)\|\leqslant e^{\alpha Mt/2}$, and similarly

$$\|g(\zeta, t, -B_-)f(\zeta, t, A_+)\|\leqslant e^{\alpha Mt/2},$$

for $0\leqslant t<\tau$. It follows with $\gamma=\alpha M$ and $C=\sup\{g(s, \lambda)^z : 0\leqslant s<\tau, \lambda\in \boldsymbol{R}\}$ that

$$\begin{aligned}
&\|U(\zeta, t/n)^n\|\leqslant\|f(\zeta, t/n, A_+)\|\\
&\quad\cdot[\|f(\zeta, t/n, -A_-)g(\zeta, t/n, B_+)\|\,\|g(\zeta, t/n, -B_-)f(\zeta, t/n, A_+)\|]^{n-1}\\
&\quad\cdot\|f(\zeta, t/n, -A_-)g(\zeta, t/n, B_+)\|\,\|g(\zeta, t/n, -B_-)\|\leqslant Ce^{\gamma t},\\
&\qquad\qquad 0\leqslant t\leqslant T,\qquad n>T/\tau.
\end{aligned}$$

III. $$U(\zeta, t/n)^n \underset{s}{\longrightarrow} \exp[-t(A_\zeta\dot{+}B_\zeta)]P,\quad n\to\infty,\ t>0,\ \zeta<0. \tag{15}$$

Here the convergence is in the same sense as in the statement of the theorem, and $A_\zeta=A_+-\zeta A_-$, $B_\zeta=B_+-\zeta B_-$.

To show convergence for $u\in\overline{\mathcal{D}}$, by Chernoff's theorem [1, Theorem 1.1], it suffices to prove that $[1+t^{-1}(1-U(\zeta, t))]^{-1}\underset{s}{\longrightarrow}[1+(A_\zeta\dot{+}B_\zeta)]^{-1}P$, $t\downarrow 0$. This, however, can be shown by the same method as in Kato [5] if we note with the conditions (ii) and (iii) of $\mathcal{F}(\tau)$ that

$$0\leqslant f(\zeta, t, A)\leqslant 1,\qquad 0\leqslant t<\tau,$$

$$[1-f(\zeta, t, A)]^{1/2} \xrightarrow[s]{} 0, \quad 1-f(\zeta, t, A)^{1/2} \xrightarrow[s]{} 0, \quad t \downarrow 0,$$

$$t^{-1/2}[1-f(\zeta, t, A)]^{1/2}u \longrightarrow A_\zeta^{1/2}u, \quad t \downarrow 0, \quad u \in \mathcal{D}(|A|^{1/2}),$$

and similarly for $g(\zeta, t, B)$. For convergence for $u \perp \mathcal{D}$, the same argument as in Kato [4] is valid.

IV. It can be seen by (1) that, for ζ with $\mathrm{Re}\,\zeta < z$, the family of the quadratic forms

$$u \mapsto \|A_+^{1/2}u\|^2 + \|B_+^{1/2}u\|^2 - \zeta\|A_-^{1/2}u\|^2 - \zeta\|B_-^{1/2}u\|^2, \quad u \in \mathcal{D}, \tag{16}$$

is holomorphic of type (a) (Kato [7, Chap. 7, § 4]). Therefore for each fixed $t \geqslant 0$ and $u \in \mathcal{H}$, $\exp[-t(A_\zeta \dotplus B_\zeta)]Pu$ is holomorphic in ζ, $\mathrm{Re}\,\zeta < z$, where $A_\zeta \dotplus B_\zeta$ denotes the m-sectorial operator in the Hilbert space $\overline{\mathcal{D}}$ associated with (16).

V. It has been seen in I and II that, for each $u \in \overline{\mathcal{D}}$, the functions $U(\zeta, t/n)^n u$ are uniformly bounded and holomorphic in ζ, $\mathrm{Re}\,\zeta < z$, as $\mathcal{B}([0, T], \mathcal{H})$-valued functions. And this sequence converges to $\exp[-t(A_\zeta \dotplus B_\zeta)]Pu$ as $n \to \infty$ for $\zeta < 0$. Therefore, by virtue of Vitali's theorem, we obtain (15) for all ζ with $\mathrm{Re}\,\zeta < z$, and in particular, the desired result (4) with $\zeta = 1$ when applied to $u \in \overline{\mathcal{D}}$. For $u \perp \mathcal{D}$, apply Vitali's theorem to the $U(\zeta, t/n)^n u$ as $\mathcal{B}([T', T], \mathcal{H})$-valued functions.

Proof of Theorem 2. For each $f_j(t, \lambda)$, define $f_j(\zeta, t, \lambda)$ as in (14) and $A_{j,\zeta} = A_{j,+} - \zeta A_{j,-}$. Set $U(\zeta, t) = f_m(\zeta, t, A_m) \cdots f_1(\zeta, t, A_1)$. Then the same arguments as in the proof of Theorem 1 apply to $U(\zeta, t/n)^n u$, with $u \in \overline{\mathcal{D}}$, except for the proof of $U(\zeta, t/n)^n u \to e^{-tC_\zeta}u$, $n \to \infty$, $t > 0$, $\zeta \in I$. Here $C_\zeta = A_{1,\zeta} \dotplus \cdots \dotplus A_{m,\zeta}$. To show this, put for each fixed $x \in \mathcal{H}$, $y_0(t) = [1 + t^{-1}(1 - U(\zeta, t))]^{-1}x$, $y_j(t) = f_j(\zeta, t, A_j)y_{j-1}(t)$, $0 < t < \tau$, $j = 1, \cdots, m$. In view of Chernoff's theorem, we have only to show that $y_0(t) \to [1 + C_\zeta]^{-1}Px$, $t \downarrow 0$. Here P denotes the orthogonal projection of $\mathcal{H}$ onto $\overline{\mathcal{D}}$. We shall use the method in Kato-Masuda [8].

Since $\|y_j(t)\| \leqslant \|x\|$ for $0 < t < \tau$, there exists a sequence $t_\nu \downarrow 0$ and $y_0^* \in \mathcal{H}$ such that $y_j(t_\nu) \xrightarrow[w]{} y_0^*$, $\nu \to \infty$. Put $\Phi_{j,\zeta}(v) = 2^{-1}\|A_{j,\zeta}^{1/2}v\|^2$ if $v \in \mathcal{D}(|A_j|^{1/2})$ and $= \infty$ otherwise. Put

$$\Phi_{j,\zeta}(t; v) = 2^{-1}\|(A_{j,+} + \zeta A_j E_j([-\delta/t, 0)))^{1/2}v\|^2$$

if $v \in \mathcal{D}(A_{j,+}^{1/2})$ and $= \infty$ otherwise. Then (11) yields, for $\zeta \in I$ and $v \in \mathcal{H}$,

$$\sum_{j=1}^{m} \Phi_{j,\zeta}(v) \geqslant \sum_{j=1}^{m} \Phi_{j,\zeta}(t_\nu; y_j(t_\nu)) + \mathrm{Re}(v - y_0(t_\nu), x - y_0(t_\nu)) + 2^{-1}t_\nu\|x - y_0(t_\nu)\|^2. \tag{17}$$

Each $\Phi_{j,\zeta}(t; y)$ is weakly lower semicontinuous in y and monotone decreasing in t, so that $\limsup_{\nu\to\infty}\Phi_{j,\zeta}(t_\nu; y_j(t_\nu)) \geqslant \sup_{t>0}\limsup_{\nu\to\infty}\Phi_{j,\zeta}(t; y_j(t_\nu)) \geqslant \sup_{t>0}\Phi_{j,\zeta}(t; y_0^*)$. It follows from (17) with $\nu \to \infty$ that

$$\sum_{j=1}^{m} \Phi_{j,\zeta}(v) \geqslant \sum_{j=1}^{m} \Phi_{j,\zeta}(y_0^*) + \mathrm{Re}(v - y_0^*, x - y_0^*).$$

This proves $y_0(t) \xrightarrow[w]{} y_0^* = [1 + C_\zeta]^{-1}Px$, $t \downarrow 0$. Hence $y_0^* \in \mathcal{D}$. Strong convergence will also be proved as in [8].

3. **Applications.** Let $V(x)$ be a real-valued measurable function on $\boldsymbol{R}^l$. Set $V_+(x)=\max\{V(x),0\}$ and $V_-(x)=\max\{-V(x),0\}$. The following facts are direct consequences of Theorem 1, although it can also be shown by the very Trotter product formula proved in Kato [5] plus the Trotter-Kato theorem [7, Chap. 9, § 2]: 1° Assume that $H^1(\boldsymbol{R}^l)\cap\mathscr{D}(V_+^{1/2})$ is dense in $L^2(\boldsymbol{R}^l)$ and V_- is form-bounded with respect to $-\Delta$ with relative bound <1 (For such V, see e.g. Faris [2]). Then $e^{-((-\Delta)\dot{+}V)}$ is positivity preserving. In fact, the approximants in (4) with $A=-\Delta$, $B=V$ and the functions (5) as f, g are all positivity preserving. 2° Let B be the same self-adjoint realization of the formal Schrödinger operator $T=-(\nabla-ib(x))^2$ as in Kato [6]. Assume that $V_+\in L^1_{loc}(\boldsymbol{R}^l)$ and V_- is form-bounded with respect to both $-\Delta$ and B with relative bounds <1. Then B obeys pointwise domination $|e^{-t(B\dot{+}V)}v|\leqslant e^{-t[(-\Delta)\dot{+}(-V_-)]}|v|$, a.e. on $\boldsymbol{R}^l$, $t\geqslant 0$, for $v\in L^2(\boldsymbol{R}^l)$.

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