110. *Optimal Economic Growth with Infinite Planning Time Horizon*

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1. Introduction. In the previous paper [5], the author established a sufficient condition which assures the existence of an optimal economic growth path in the case of *finite* planning time horizon. However it is quite apparent that we must encounter with various difficulties if we try to extend our analysis to the case of *infinite* time horizon. The purpose of the present paper is to show the way to overcome such difficulties. The key point is to transform the problem with an infinite measure space to the equivalent one with a finite measure space by means of the weighted Sobolev space. The author is indebted to Berkovitz [1] and Chichilnisky [2] for the important ideas embodied in the proof.

2. Problem. Let us recapitulate the notations which are slightly different from those used in [5].

$\boldsymbol{R}_+=[0,\infty)$ planning time horizon.

$u:\boldsymbol{R}_+\times\boldsymbol{R}^l_+\to\boldsymbol{R}_+$ welfare function at each time.

$f:\boldsymbol{R}_+\times\boldsymbol{R}^l_+\to\boldsymbol{R}^l_+$ production function at each time.

$\delta>0$ the discount rate of the welfare in the future.

$\lambda\in(0,1)^l$ the vector of the depreciation rates of l capital goods.

Furthermore we have a couple of variable mappings to be optimized;

$k:\boldsymbol{R}_+\to\boldsymbol{R}^l_+$ path of capital accumulation.

$s:\boldsymbol{R}_+\to[0,1]^l$ path of the vector whose components are saving rates of each goods.

For any vector $x\in\boldsymbol{R}^l$, we designate by M_x the diagonal matrix of the form

$$M_x=\begin{pmatrix} x_1 & & & 0 \\ & x_2 & & \\ & & \ddots & \\ 0 & & & x_l \end{pmatrix}$$

where x_i $(1\leqq i\leqq l)$ is the i-th coordinate of x.

Then the problem of optimal economic growth can be formulated as follows:

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(P)

Maximize

$$J(k,s)=\int_0^{\infty} u[t,(I-M_{s(t)})f(t,k(t))]e^{-\delta t}dt \tag{1}$$

subject to

$$\dot{k}(t)=M_{s(t)}f(t,k(t))-M_{\lambda}k(t) \tag{2}$$

$$k(0)=\bar{k}\ \text{(given vector)}. \tag{3}$$

(I is the identity matrix.)

Define $w: \boldsymbol{R}_+\times\boldsymbol{R}_+^l\times[0,1]^l\to\boldsymbol{R}_+$ and $g: \boldsymbol{R}_+\times\boldsymbol{R}_+^l\times[0,1]^l\to\boldsymbol{R}^l$ by

$$w(t,k,s)=u[t,(I-M_s)f(t,k)]$$

and

$$g(t,k,s)=M_sf(t,k)-M_{\lambda}k$$

respectively. Furthermore let ν be a finite measure on $\boldsymbol{R}_+$ defined by

$$\nu(E)=\int_E e^{-\delta t}dt$$

for every Lebesgue-measurable set E in $\boldsymbol{R}_+$. Then the problem (P) can be rewritten in the form:

(P′)

Maximize

$$J(k,s)=\int_0^{\infty} w(t,k(t),s(t))d\nu \tag{1′}$$

subject to

$$\dot{k}(t)=g(t,k(t),s(t)) \tag{2′}$$

$$k(0)=\bar{k}. \tag{3′}$$

Throughout this paper, we shall assume the following conditions to be satisfied.

Assumption 1. u is continuous on $\boldsymbol{R}_+\times\boldsymbol{R}_+^l$ and concave in the last l-dimensional vector.

Assumption 2. f is continuous on $\boldsymbol{R}_+\times\boldsymbol{R}_+^l$.

Assumption 3. There exists $C>0$ such that

$$k_i\geqq C \quad \text{implies} \quad \sup_{t\in\boldsymbol{R}_+} f_i(t,k)\leqq\lambda_i k_i$$

for any $i(=1,2,\cdots,l)$, where k_i (resp. f_i) is the i-th coordinate of k (resp. f).

Assumption 4. There exists a couple of positive constants, α and β, such that

$$0<\beta<\delta/2 \text{ and } \|f(t,k)\|<\alpha\|k\|\,e^{\beta t} \qquad \text{for all } t\in\boldsymbol{R}_+ \text{ and } k\in\boldsymbol{R}_+^l.$$

Assumption 5. There exist a non-negative ν-integrable function $\theta:\boldsymbol{R}_+\to\boldsymbol{R}$ and a vector $b\in\boldsymbol{R}^l$ such that

$$w(t,k,s)-\langle b,g(t,k,s)\rangle\leqq\theta(t) \qquad \text{for every } (t,k,s).$$

3. Boundedness of admissible paths. We denote by $\mathcal{S}$ the set of all the measurable mappings $s:\boldsymbol{R}_+\to[0,1]^l$, and we also denote by $\mathcal{W}_\delta^{1,2}$ the weighted Sobolev space on $\boldsymbol{R}_+$ with the weight function $e^{-\delta t}$. (Cf. Kufner [3] or Kufner *et al.* [4, pp. 417–423].)

Definition. A pair (k,s) $\mathcal{W}_\delta^{1,2}\times\mathcal{S}$ is said to be an *admissible pair* if it satisfies (2) and (3). And $k\in\mathcal{W}_\delta^{1,2}$ is called an *admissible path* if there exists an $s \in \mathcal{S}$ such that (k, s) is an admissible pair. The set of all the admissible pairs is denoted by $\mathcal{A}$, and the set of all the admissible paths by $\mathcal{A}_k$.

Thanks to Assumption 3, it can be proved in the same manner as in Proposition 1 of [5] that

$$\sup_{k \in \mathcal{A}_k} \|k\|_{\infty,\nu} < lC. \tag{4}$$

Furthermore, taking account of Assumption 4, we have

$$\begin{aligned} \|\dot{k}(t)\| &\leqq \|M_{s(t)} f(t, k(t))\| + \|M_\lambda k(t)\| \\ &\leqq \alpha \|k(t)\| e^{\beta t} + lC \\ &\leqq lC(\alpha e^{\beta t} + 1). \end{aligned} \tag{5}$$

The right-hand side is ν-integrable. Hence

$$\sup_{k \in \mathcal{A}_k} \|\dot{k}\|_{1,\nu} \leqq N_1 \qquad \text{for some } 0 < N_1 < \infty. \tag{6}$$

Similarly

$$\sup_{k \in \mathcal{A}_k} \|\dot{k}\|_{2,\nu} \leqq N_2 \qquad \text{for some } 0 < N_2 < \infty. \tag{7}$$

Proposition 1. *$\mathcal{A}_k$ is bounded in $\mathcal{W}_\delta^{1,2}$.*

Corollary 1. *$\mathcal{A}_k$ is weakly sequentially compact in $\mathcal{W}_\delta^{1,2}$.*

Proof. Since $\mathcal{W}_\delta^{1,2}$ is a Hilbert space, the boundedness of $\mathcal{A}_k$ implies that it is weakly sequentially compact. Q.E.D.

4. Existence Theorem. **Proposition 2.** *$\gamma \equiv \sup_{(k,s) \in \mathcal{A}} J(k, s)$ is finite.*

Proof. Let $\{k_n, s_n\}$ be a sequence in $\mathcal{A}$ such that

$$\lim_n J(k_n, s_n) = \gamma.$$

By Assumption 5,

$$\begin{aligned} &w(t, k_n(t), s_n(t)) \\ &\qquad \leqq \theta(t) + \langle b, g(t, k_n(t), s_n(t)) \rangle \\ &\qquad = \theta(t) + \langle b, \dot{k}_n(t) \rangle. \end{aligned}$$

Taking account of (6), we obtain

$$\begin{aligned} &\int_0^\infty w(t, k_n(t), s_n(t)) d\nu \\ &\qquad \leqq \int_0^\infty \theta(t) d\nu + \int_0^\infty \langle b, \dot{k}_n(t) \rangle d\nu \\ &\qquad \leqq \int_0^\infty \theta(t) d\nu + N_1 \|b\| \\ &\qquad < \infty \qquad \text{for all } n. \end{aligned} \tag{8}$$

Thus we can conclude that γ must be finite. Q.E.D.

Let us define the correspondence (=multi-valued mapping) $\Omega : \boldsymbol{R}_+ \times \boldsymbol{R}_+^l \twoheadrightarrow \boldsymbol{R}^l \times \boldsymbol{R}_+$ by

$$\begin{aligned} \Omega(t, k) = \{(\xi, \eta) \in \boldsymbol{R}^l \times \boldsymbol{R}_+ \mid \xi = g(t, k, s) \quad &\text{and} \\ 0 \leqq \eta \leqq w(t, k, s) \qquad &\text{for some } s \in [0, 1]^l\}. \end{aligned} \tag{9}$$

Thanks to our Assumptions 1 and 2, it is quite easy to prove that Ω is a compact-convex-valued continuous correspondence. Therefore the correspondence

$$k \twoheadrightarrow \Omega(\tilde{t}, k) = \overline{\mathrm{co}}\, \Omega(\tilde{t}, k) \tag{10}$$

is also a compact-convex-valued continuous correspondence for each fixed $\tilde{t} \in \boldsymbol{R}_+$. If we denote

$$K(\tilde{t}\,;\tilde{k},\varepsilon) = \{(\tilde{t}, k) \in \boldsymbol{R}_+ \times \boldsymbol{R}^l_+ \mid \|k - \tilde{k}\| < \varepsilon\}$$

$((\tilde{t}, \tilde{k}) \in \boldsymbol{R}_+ \times \boldsymbol{R}^l_+)$, then we obtain the following result as a consequence of the continuity of the correspondence (10).

Proposition 3. *For each* $(\tilde{t}, \tilde{k}) \in \boldsymbol{R}_+ \times \boldsymbol{R}^l_+$,

$$\Omega(\tilde{t}, \tilde{k}) = \bigcap_{\varepsilon > 0} \overline{\mathrm{co}}\, \Omega(k(\tilde{t}\,;\tilde{k}, \varepsilon)).$$

Let $\{(k_n, s_n)\}$ be a sequence in $\mathcal{A}$ such that

$$\lim_n J(k_n, s_n) = \gamma. \tag{11}$$

Then, by Corollary 1, there exists a weakly convergent subsequence (no change in notations) of $\{k_n\}$; i.e.

$$k_n \longrightarrow k^* \qquad \text{weakly in } \mathcal{W}^{1,2}_\delta. \tag{12}$$

Thus we have just finished up the preparation for the following crucial proposition, the proof of which can be carried out almost in the same manner as in Proposition 4 of Maruyama [5] except for some technical details.

Proposition 4. *There exists a ν-integrable function* $\zeta : \boldsymbol{R}_+ \longrightarrow \boldsymbol{R}$ *such that*

$$\int_0^\infty \zeta(t) d\nu \geqq \gamma \tag{13}$$

and

$$(k^*(t), \zeta(t)) \in \Omega(t, k^*(t)) \qquad a.e. \tag{14}$$

By Proposition 4, it has been verified that the value

$$\int_0^\infty \zeta(t) d\nu$$

can be attained under the path $k^*(t)$ if $s(t) \in [0,1]^l$ is suitably chosen at each $t \in \boldsymbol{R}_+$. Finally we have to prove that $s(t)$ can be chosen so as to be measurable. We can achieve this object through a similar argument as in [5] if we simply replace $([0, T], dt)$ by $(\boldsymbol{R}_+, \nu)$. Thus we have

Theorem. *Under Assumptions* 1–5, *the problem* (P) *has an optimal solution.*

References

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