88. On the Galois Cohomology Groups of C_K/D_K

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1. Let k be an algebraic number field and K be its finite Galois extension of degree n with the group G. We denote by C_K and D_K the idele class group of K and its connected component of the unity respectively. In this note, we shall determine the structure of the cohomology group $H^p(G, C_K/D_K)$ for non-negative integer p. For cohomology groups and the morphisms concerned with them, we shall use the notation and terminology as is given in S. Iyanaga [3].

2. In this section, p denotes an arbitrary integer. Let us denote the idele group of K by J_K and its connected component of the unity by H_K. We denote by E the set of all imaginary places of K. Then the maximal compact subgroup of H_K is given by $H'_K=\{x=(x_\mathfrak{p})\in J_K \mid x_\mathfrak{p}=1 \text{ if } \mathfrak{p}\notin E, |x_\mathfrak{p}|=1 \text{ if } \mathfrak{p}\in E\}$. Let us denote the canonical homomorphism from J_K to C_K by φ and $\varphi(H'_K)$ by D'_K. Then we have the following exact sequence

$$(1) \qquad 1\longrightarrow H'_K \xrightarrow{\varphi} C_K \xrightarrow{\psi} C_K/D'_K \longrightarrow 1.$$

By cohomology sequences belonging to (1) and the fact that $H^q(G, H'_K)=0$ if q is odd, we have

$$(2) \qquad \begin{aligned} 0&\longrightarrow H^{2p+1}(C_K)\longrightarrow H^{2p+1}(C_K/D'_K)\longrightarrow H^{2p+2}(H'_K) \\ &\longrightarrow H^{2p+2}(C_K)\longrightarrow H^{2p+2}(C_K/D'_K)\longrightarrow 0 \qquad \text{(exact).} \end{aligned}$$

Here we have abbreviated $H^q(G, A)$ to $H^q(A)$ for a G-module A.

Since D_K/D'_K is uniquely divisible, we obtain the isomorphism

$$(3) \qquad H^p(G, C_K/D_K)\cong H^p(G, C_K/D'_K).$$

Hereafter, by virtue of (3), we shall only be concerned with the determination of $H^p(G, C_K/D'_K)$ instead of $H^p(G, C_K/D_K)$.

Let $\{\mathfrak{p}_i \mid 1\leqq i\leqq r\}$ be the set of all real places of k which ramify in K. If $r=0$, it follows from (2) that

$$H^p(G, C_K/D'_K)\cong H^p(G, C_K)\cong H^{p-2}(G, Z).$$

Therefore, in the following, we exclude this case and shall treat only the case $r>0$. This implies that n is even, so we put $m=n/2\in Z$.

Let $\mathfrak{P}_i$ be one of the extensions of $\mathfrak{p}_i$ to K, and N_i be the decomposition group of $\mathfrak{P}_i$. Let us denote the transfer homomorphism from N_i to G and the restriction from G to N_i on cohomology groups by $\tau^{N_i,G}$ and ρ^{G,N_i} respectively. Since $H^{2p}(G, H'_K)$ is generated by $\tau^{N_i,G}H^{2p}(N_i, H'_K)$, we obtain the following lemma.

Lemma 1. *Under the above notation, we have*

$$H^{2p}(G, H'_K)=\langle \tau^{N_i,G}H^{2p}(N_i, H'_K)\,|\,1\leqq i\leqq r\rangle \cong Z_2^r.$$

Here $\langle \tau^{N_i,G}H^{2p}(N_i, H'_K)\,|\,1\leqq i\leqq r\rangle$ *denotes the group generated by all* $\tau^{N_i,G}H^{2p}(N_i, H'_K)$ *and* Z_2^r *denotes the elementary abelian group of order* 2^r.

The following theorem can be proved by using Lemma 1 and (2).

Theorem 1. *For all* $p\in Z$, *we have the isomorphism*

$$H^{2p}(G, C_K/D'_K)\cong H^{2p-2}(G, Z)/\langle \tau^{N_i,G}H^{2p-2}(N_i, Z)\,|\,1\leqq i\leqq r\rangle.$$

Corollary 1. *Let* N *denote the group generated by* $N_1, \cdots, N_r$. *Then we have*

$$H^0(G, C_K/D'_K)\cong G/[G, G]N.$$

3. Hereafter, we shall write $H^q(G, Z)$ as an additive group. If G is commutative, we can easily show that $H^{2p}(N_i, Z)=\rho^{G,N_i}H^{2p}(G, Z)$ for all $p\geqq 0$. Hence it immediately follows that $\tau^{N_i,G}H^{2p}(N_i, Z)=mH^{2p}(G, Z)$. Here we denote $mH^{2p}(G, Z)=\langle mx\,|\,x\in H^{2p}(G, Z)\rangle$. Therefore, if G is commutative, we obtain the following isomorphism for all $p\geqq 0$.

$$(4)\qquad H^{2p+2}(G, C_K/D'_K)\cong H^{2p}(G, Z)/mH^{2p}(G, Z).$$

By virtue of (2) and (4), using cup product, we have

$$(5)\qquad H^{2p+1}(G, C_K/D'_K)\cong H^{2p-1}(G, Z)\times M,$$

where M is isomorphic to Z_2^{r-1} (resp. Z_2^r) if the 2-Sylow subgroup of G is cyclic (resp. not cyclic).

Lemma 2. *Let* 2^l *be the highest power of* 2 *dividing* n, *and* S *be a* 2*-Sylow subgroup of* G. *We denote* 2^{l-1} *by* t. *Then* $t\in Z$ *and for all* $p\geqq 0$, *we have*

$$H^{2p+2}(G, C_K/D'_K)\cong H^{2p}(G, Z)/mH^{2p}(G, Z),$$

if and only if $H^{2p+2}(S, C_K/D'_K)\cong H^{2p}(S, Z)/tH^{2p}(S, Z)$.

Lemma 3. *Let* G *be a generalized quaternion group of order* 2^l. *We denote* 2^{l-1} *by* t. *Then for all* $p\in Z$, *we have*

$$H^{4p}(G, C_K/D'_K)\cong Z_2^2,$$
$$H^{4p+1}(G, C_K/D'_K)\cong Z_2^{r-1},$$
$$H^{4p+2}(G, C_K/D'_K)\cong Z_t,$$
$$H^{4p+3}(G, C_K/D'_K)\cong Z_2^r.$$

A detailed proof of the above lemmas is given in [4].

Theorem 2. *Let the notation be as above. For all* $p\geqq 0$, *we have*

$$H^{2p+2}(G, C_K/D'_K)\cong H^{2p}(G, Z)/mH^{2p}(G, Z).$$

Proof. By Lemma 2, we may assume that G is a 2-group. If G is neither cyclic group nor a generalized quaternion group, we can find a subgroup L_i of G, for every N_i, such that $L_i\supset N_i$ and $L_i\cong Z_2^2$. From Theorem 1 and (4), it follows that $2H^{2p}(L_i, Z)=0$. By virtue of the associativity of the transfer homomorphism, we have $mH^{2p}(G, Z)=\langle \tau^{N_i,G}H^{2p}(N_i, Z)\,|\,1\leqq i\leqq r\rangle=0$. Then our assertion is the immediate consequence of Theorem 1.

If G is either cyclic or a generalized quaternion, our theorem follows from (4) and Lemma 3.

Corollary 2. *Let S be a 2-Sylow subgroup of G. Then for all $p\geqq 0$, we have*

$$H^{2p+1}(G, C_K/D'_K)\cong H^{2p-1}(G, Z)\times M,$$

where M is classified as follows,

i) $M\cong Z_2^{r-1}$ *for the following three cases,*

a) $p=0$,

b) *S is cyclic,*

c) *p is even and S is a generalized quaternion.*

ii) $M\cong Z_2^r$ *for other cases.*

Proof. We can easily verify that $mH^{2p}(G, Z)=Z_2$ in the cases a), b) and c) and that $mH^{2p}(G, Z)=0$ otherwise. Therefore, using cup product, our conclusion follows from Theorem 2 and (2).

References

[1] E. Artin and J. Tate: Class Field Theory. Benjamin, New York (1967).

[2] H. Cartan and S. Eilenberg: Homological Algebra. Princeton Univ. Press, Princeton, N. J. (1956).

[3] S. Iyanaga (ed.): The Theory of Numbers. North Holland/American Elsevier (1975).

[4] S. Katayama: On the Galois cohomology groups of C_K/D_K. Master Thesis, Kyoto Univ. (1981).

[5] R. C. Lyndon: Cohomology theory of group extensions. Duke Math. J., **15**, 271–292 (1948).

[6] R. G. Swan: The p-period of a finite group. Illinois J. Math., **4**, 341–346 (1960).