77. *On Ranked Linear Spaces. II*

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We shall explain in this note in detail the completion of the ranked linear spaces defined in I, as mentioned in I, § 1. The references here are the same as those in I*).

§ 5. **Completion of ranked linear spaces. Definition.** Let $(E, \mathfrak{V}_{E,n}, \mathfrak{V}_{E,n}^{(p)})$ be a separated ranked linear space and $(\hat{E}, \hat{\mathfrak{V}}_{\hat{E},n}, \hat{\mathfrak{V}}_{\hat{E},n}^{(\hat{p})})$ a complete ranked linear space. $(\hat{E}, \hat{\mathfrak{V}}_{\hat{E},n}, \hat{\mathfrak{V}}_{\hat{E},n}^{(\hat{p})})$ is said to be a *completion* of $(E, \mathfrak{V}_{E,n}, \mathfrak{V}_{E,n}^{(p)})$ if $(E, \mathfrak{V}_{E,n}, \mathfrak{V}_{E,n}^{(p)})$ is equivalent to a ranked linear subspace of $(\hat{E}, \hat{\mathfrak{V}}_{\hat{E},n}, \hat{\mathfrak{V}}_{\hat{E},n}^{(\hat{p})})$ which is dense in $(\hat{E}, \hat{\mathfrak{V}}_{\hat{E},n}, \hat{\mathfrak{V}}_{\hat{E},n}^{(\hat{p})})$.

We shall now construct a completion of a given separated ranked linear space $(E, \mathfrak{V}_{E,n}, \mathfrak{V}_{E,n}^{(p)})$.

Let us denote by M' the family of canonical fundamental sequences in $(E, \mathfrak{V}_{E,n}, \mathfrak{V}_{E,n}^{(p)})$. We introduce in M' an equivalence relation ρ defined as follows:

For $u, v \in M'$, $u\rho v$ iff there exists a $w \in M'$ satisfying $u \prec w$ and $v \prec w$. (This is the same equivalence relation as that used in [7] and [9].) Remark that for two $u=\{p_i+U_i\}$ and $v=\{q_i+V_i\}$ in M' it holds that $u\rho v$ iff $\{p_i-q_i\}\to 0$.

Then we have

Lemma 1. *If* $u\cap v\neq\phi$ *for* $u, v\in M'$, *then* $u\rho v$. (Here $u\cap v\neq\phi$ means $(p_i+U_i)\cap(q_i+V_i)\neq\phi$ for all i where $u=\{p_i+U_i\}$ and $v=\{q_i+V_i\}$.)

Furthermore,

Lemma 2. *For any* $u=\{p_i+U_i\}$, $v=\{q_i+V_i\}$ *in* M' *and any scalar* $\alpha\neq 0$, *there exist* w *and* w' *in* M' *such that* $u+v\prec w$ *and* $\alpha u\prec w'$. (Here $u+v$ and αu mean the sequences $\{p_i+U_i+q_i+V_i\}$ and $\{\alpha p_i+\alpha U_i\}$ respectively.)

By virtue of Lemma 2, we can define linear operations in $\tilde{M}=M'/\rho$. (Hereafter we shall denote by $\tilde{u}$ the equivalence class that include $u\in M'$.) For $\tilde{u}, \tilde{v}\in\tilde{M}$ and a scalar $\alpha\neq 0$, there exist w and w' in M' such that $u+v\prec w$ and $\alpha u\prec w'$ by Lemma 2, we define $\tilde{w}=\tilde{u}+\tilde{v}$ and $\tilde{w}'=\alpha\tilde{u}$.

Let τ be a mapping of E into $\tilde{M}$ such that $\tau(p)=\tilde{u}_p$, where u_p is a p-canonical fundamental sequence.

Lemma 3. *The mapping* $\tau: E\to\tau(E)$ *is linear and one-to-one.*

*) Teruko Tsuda, On ranked linear spaces I. Proc. Japan Acad., **57**A, 262–266 (1981).

Obviously $u \in \tilde{u} \in \tilde{M} \backslash \tau(E)$ *implies* $\theta(u)=\bigcap_{i=0}^{\infty}(p_i+U_i)=\phi$ *where* $u=\{p_i+U_i\}$.

Let $\hat{E}=E \cup (\tilde{M} \backslash \tau(E))$ have the linearity induced from M'.

Let us denote with $(p+U)^{\wedge}$ the union of $p+U$ $(\subset E)$ with the subset of $\tilde{M} \backslash \tau(E)$ consisting of the classes $\tilde{u}$ such that u starts with the first term $p+U$.

Lemma 4. *If* $U(0,n) \supset V(0,m)$ *and* $n<m$, *then* $(U(0,n))^{\wedge} \supset (V(0,m))^{\wedge}$, *and for any* $U, V \in \mathfrak{V}_E(0)$, $(U)^{\wedge} \supset (V)^{\wedge}$ *implies* $U \supset V$.

We now define a family of pre-neighborhoods in $\hat{E}$ as follows:

Let $\hat{p} \in \hat{E}$. If $\hat{p}=p \in E$, we define $\hat{\mathfrak{V}}_{\hat{E},n}(\hat{p})=\{p+(U)^{\wedge} : U \in \mathfrak{V}_{E,6n}^{(p)}(0)\}$, and if $\hat{p} \notin E$, we define $\hat{\mathfrak{V}}_{\hat{E},n}(\hat{p})$ as the set of $\hat{p}+(U)^{\wedge}$, such that an element u of $\hat{p}$ starts with the first term $r+U$ with $U \in \mathfrak{V}_{E,6n}^{(r)}(0)$. Let $\hat{\mathfrak{V}}_{\hat{E}}(\hat{p})=\bigcup_n \hat{\mathfrak{V}}_{\hat{E},n}(\hat{p})$.

Then, if $\hat{p}=p \in E$, $\hat{\mathfrak{V}}_{\hat{E},n}^{(\hat{p})}(0)=\{(U)^{\wedge} : U \in \mathfrak{V}_{E,6n}^{(p)}(0)\}$, and if $\hat{p} \notin E$, $\hat{\mathfrak{V}}_{\hat{E},n}^{(\hat{p})}(0)$ is the set of $(U)^{\wedge}$, such that an element u of $\hat{p}$ starts with the first term $r+U$ with $U \in \mathfrak{V}_{E,6n}^{(r)}(0)$.

Lemma 5. *If* $U_1(0,n_1) \subset U_2(0,n_2)$ *and* $n_2+2<n_1$, *then for any point* $p \in E$ *such that* $U_1 \in \mathfrak{V}_E^{(p)}(0)$, *we have* $(p+U_1)^{\wedge} \subset p+(U_2)^{\wedge}$ *and* $p+(U_1)^{\wedge} \subset (p+U_2)^{\wedge}$.

From these lemmas, we obtain

Proposition 1. *The above system of pre-neighborhoods has the properties* (1)–(4), (I)–(III) *in* I, § 2; *therefore* $(\hat{E}, \hat{\mathfrak{V}}_{\hat{E},n}, \hat{\mathfrak{V}}_{\hat{E},n}^{(\hat{p})})$ *is a ranked linear space.*

Proposition 2. *Define* $\mathfrak{V}'_{E,n}$ *as the set of intersections of elements of* $\hat{\mathfrak{V}}_{\hat{E},n}$ *with* E, *and* $\mathfrak{V}'^{(p)}_{E,n}$ *as the set of intersections of elements of* $\hat{\mathfrak{V}}_{\hat{E},n}^{(p)}$ *with* E. *Then* $(E, \mathfrak{V}'_{E,n}, \mathfrak{V}'^{(p)}_{E,n})$ *becomes a ranked linear subspace of* $(\hat{E}, \hat{\mathfrak{V}}_{\hat{E},n}, \hat{\mathfrak{V}}_{\hat{E},n}^{(\hat{p})})$ *which is equivalent with* $(E, \mathfrak{V}_{E,n}, \mathfrak{V}_{E,n}^{(p)})$ *and is dense in* $(\hat{E}, \hat{\mathfrak{V}}_{\hat{E},n}, \hat{\mathfrak{V}}_{\hat{E},n}^{(\hat{p})})$.

We now introduce the following definition in order to facilitate the formulation of the next lemma.

Definition. Let $p+U$ be a pre-neighborhood of point p in a ranked linear space $(E, \mathfrak{V}_{E,n}, \mathfrak{V}_{E,n}^{(p)})$. The point p will be called the *center* of this pre-neighborhood. If all the centers of terms of a fundamental sequence $\hat{u}$ in $(\hat{E}, \hat{\mathfrak{V}}_{\hat{E},n}, \hat{\mathfrak{V}}_{\hat{E},n}^{(\hat{p})})$ are in E, $\hat{u}$ is called *E-centered.*

Lemma 6. *For any canonical fundamental sequence* $\hat{u}$ *in* $\hat{E}$, *there exists an E-centered canonical fundamental sequence* $\hat{v}$ *in* $\hat{E}$ *such that* $\hat{u} \sim \hat{v}$.

By this lemma, we can prove

Proposition 3. $(\hat{E}, \hat{\mathfrak{V}}_{\hat{E},n}, \hat{\mathfrak{V}}_{\hat{E},n}^{(\hat{p})})$ *is complete.*

Thus we have established

Theorem 3. *For any separated ranked linear space* $(E, \mathfrak{V}_{E,n}, \mathfrak{V}_{E,n}^{(p)})$, *we can construct a completion* $(\hat{E}, \hat{\mathfrak{V}}_{\hat{E},n}, \hat{\mathfrak{V}}_{\hat{E},n}^{(\hat{p})})$ *of* $(E, \mathfrak{V}_{E,n}, \mathfrak{V}_{E,n}^{(p)})$.

§6. **Uniqueness.** In this section we shall give a condition for the space $\hat{E}$ to be separated, and show that the separated completion of E, if it exists, is unique up to isomorphism.

Proposition 4. *Under the following condition* (S), $(\hat{E}, \hat{\mathfrak{V}}_{\hat{E},n}, \hat{\mathfrak{V}}_{\hat{E},n}^{(\hat{p})})$ *is separated.*

(S): Let $u=\{p_i+U_i(0,n_i)\}$ and $v=\{q_i+V_i(0,m_i)\}$ be canonical fundamental sequences in E. If $u\rho v$ and $W(0,l)\supset((p_0-q_0)+U_0)\cup V_0$, $l+2<\min(n_0,m_0)$, then there exists a $W'(0,\min(n_0,m_0)-2)$ such that $W\supset W'\supset((p_0-q_0)+U_0)\cup V_0$.

Proposition 5. *If* $(\hat{F}, \hat{\mathfrak{V}}_{\hat{F},n}, \hat{\mathfrak{V}}_{\hat{F},n}^{(\hat{q})})$ *and* $(\hat{G}, \hat{\mathfrak{V}}_{\hat{G},n}, \hat{\mathfrak{V}}_{\hat{G},n}^{(\hat{r})})$ *are separated completions of* $(E, \mathfrak{V}_{E,n}, \mathfrak{V}_{E,n}^{(p)})$, *then* $\hat{F}$ *and* $\hat{G}$ *are isomorphic.*

This follows from the following

Lemma 7. *Let* $(E, \mathfrak{V}_{E,n}, \mathfrak{V}_{E,n}^{(p)})$ *and* $(F, \mathfrak{V}_{F,n}, \mathfrak{V}_{F,n}^{(q)})$ *be ranked linear spaces and suppose that* E *is a ranked linear subspace of* F. *Then for any canonical fundamental sequence* $u=\{p_i+U_i\}$ *in* E, *there is a canonical fundamental sequence* $v=\{q_i+V_i\}$ *in* F *such that* $v|E\sim u$. *Furthermore we can take this* v *so that the sequence* $\{q_i\}$ *is a subsequence of the sequence* $\{p_i\}$. (Here $v|E$ means the decreasing sequence $\{(q_i+V_i)\cap E\}$ in E.)

Example. Consider again the example in I, §2 and let $(\hat{E}, \hat{\mathfrak{V}}_{\hat{E},n}, \hat{\mathfrak{V}}_{\hat{E},n}^{(\hat{p})})$, $(\hat{E}_k, \hat{\mathfrak{V}}_{\hat{E}_k,n}, \mathfrak{V}_{\hat{E}_k,n}^{(\hat{p}^{(k)})})$ $(k=1,2,\cdots)$ be completions of E, E_k respectively obtained in §5. Let for each k, $\tau_k:\hat{E}_k\to\hat{E}$ be a mapping defined by $\tau_k(\hat{p}^{(k)})=\tilde{u}^{(k)}$ where $u^{(k)}\in\hat{p}^{(k)}$ ($u^{(k)}$ is also a canonical fundamental sequence in E) for every $\hat{p}^{(k)}\in\hat{E}_k$. Then the ranked union space $\bigcup_{k=1}^{\infty}\tau_k(\hat{E}_k)$ of the sequence of ranked linear spaces $\{(\tau_k(\hat{E}_k), \tau_k(\hat{\mathfrak{V}}_{\hat{E}_k,n}), \tau_k(\hat{\mathfrak{V}}_{\hat{E}_k,n}^{(\hat{p}^{(k)})}))\}$ becomes a ranked linear space and it is equivalent to $\hat{E}$. Especially in case E is *strict union* (cf. [6]), it holds that $\tau_k(\hat{E}_k)$ and $\hat{E}_k$ are isomorphic, hence $\hat{E}$ is equivalent to $\bigcup_{k=1}^{\infty}\hat{E}_k$.

§7. **Remark on the completion of ranked spaces.** We remark that the condition (S) in §6 is equivalent to:

(S′): Let $u=\{U_i(p_i,n_i)\}$ and $v=\{V_i(q_i,m_i)\}$ be canonical fundamental sequences in $(E, \mathfrak{V}_{E,n}, \mathfrak{V}_{E,n}^{(p)})$. If $u\rho v$ and $W(p_0,l)\supset U_0\cup V_0$, $l+2<\min(n_0,m_0)$, then there exsits a $W'(p_0,\min(n_0,m_0)-2)$ such that $W\supset W'\supset U_0\cup V_0$.

Notice that this condition (S′) is stated without using linearity of $(E, \mathfrak{V}_{E,n}, \mathfrak{V}_{E,n}^{(p)})$ so that it has a meaning for any ranked spaces of indicator ω_0.

It was shown in Okano [8] that ranked spaces satisfying certain conditions can be completed. We mention that we can complete ranked (not necessarily linear) spaces of indicator ω_0 satisfying the following conditions (1)–(4) below (these can be applied to the case of separated ranked linear spaces in I, § 2) and (S′) by our constructive method as in § 5 to ranked spaces satisfying (1)–(4). We can also show that the completion satisfying (1)–(4) is unique up to isomorphism.

Let $(E, \mathfrak{B}_n)$ be a ranked space of indicator ω_0.

(1) If $U_0(p_0, n_0) \supset U_1(p_0, n_1) \supset U_2(p_2, n_2) \supset U_3(p_2, n_3) \ni q$ and $n_0 < n_1 < n_2 < n_3$, then there is a $W(q, n_3 - 1)$ such that $U_0 \supset W \supset U_3$.

(2) For $V_1(q, m_1)$ and $V_2(q, m_2)$ such that $m_1, m_2 \geq 1$, there is a $U(q, \min(m_1, m_2) - 1)$ satisfying $U \supset V_1 \cup V_2$. Further if $U_1(p, n_1) \supset U_2(p, n_2) \supset V_1(q, m_1) \cup V_2(q, m_2)$, $n_1 < n_2$, $n_2 + k < \min(m_1, m_2) (k = 1, 2, \cdots)$, then there exist $W_i(q, \min(m_1, m_2) - k + i)$ $(i = 0, 1, \cdots, k-1)$ such that $U_1 \supset W_0 \supset W_1 \supset \cdots \supset W_{k-1} \supset V_1 \cup V_2$.

(3) If $U(p, n) \supset V(p, m)$ and $n < k < m$, then there is a $W(p, k)$ such that $U \supset W \supset V$.

(4) For every p in E and every p-fundamental sequence $u_p = \{U_i(p, n_i)\}$, it holds $\theta(u_p) = \bigcap_{i=0}^{\infty} U_i(p, n_i) = \{p\}$. (Namely E is separated.)

The system of pre-neighborhoods of $\hat{E}$ is given as follows:

Let $\hat{p} \in \hat{E}$. If $\hat{p} = p \in E$, we define $\hat{\mathfrak{B}}_n(\hat{p}) = \{(U(p, 3n))^\wedge : U \in \mathfrak{B}_{3n}(p)\}$, and if $\hat{p} \notin E$, we define $\hat{\mathfrak{B}}_n(\hat{p})$ as the set of $(U(q, 3n))^\wedge$, such that an element u of $\hat{p}$ starts with the first term $U(q, 3n)$ with $U \in \mathfrak{B}_{3n}(q)$ and the rank of second term is greater than $3n+3$.

We should finally like to remark that our method of construction of $\hat{E}$ is essentially the same under the condition (S′) as that used in [8]. In fact, for any pre-neighborhoods of $\hat{p}$ in $\hat{E}$, $\hat{U}(\hat{p}, n) = (U(q, 3n))^\wedge$ and $\hat{V}(\hat{p}, m) = (V(r, 3m))^\wedge$, such that $\hat{U} \supset \hat{V}$ and $n < m$, we have u and v of $\hat{p}$ such that u and v start with the first term $U(q, 3n)$ and $V(r, 3m)$ respectively and the rank n_1 of second term $U_1(q, n_1)$ of u is greater than $3n+3$. So by (S′) there exists a $W(q, \min(n_1 - 1, 3m) - 2)$ such that $U \supset W \supset U_1 \cup V$. Hence $\hat{U}(\hat{p}, n) = (U(q, 3n))^\wedge \supset (W(q, \min(n_1 - 1, 3m) - 2))^\wedge \supset (V(r, 3m))^\wedge = \hat{V}(\hat{p}, m)$, this is just the condition (2) in [8].