75. On a Certain Decomposition of 2-Dimensional Cycles on a Product of Two Algebraic Surfaces

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In this note, we define a type of decomposition for the 4-dimensional cohomology group of a product of two algebraic surfaces and we use such a decomposition for investigation of algebraic 2-cycles on it. Details of this note will appear elsewhere.

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§ 1. Hodge-Künneth-Transcendence-decomposition. Let S and S' be non-singular projective surfaces defined over the field of complex numbers $\boldsymbol{C}$. We denote by $C^r(S\times S')$ the group of all cycles of codimension r on $S\times S'$ modulo rational equivalence, and we have a cycle map cl, which to each cycle $X\in C^r(S\times S')\otimes_{\boldsymbol{Z}}\boldsymbol{Q}$ associates the cohomology class $cl(X)\in H^{2r}(S\times S',\boldsymbol{C})$. Let $H^{2r}(S\times S',\boldsymbol{Q})_{\mathrm{alg}}$ denote the image of $cl: C^r(S\times S')\otimes_{\boldsymbol{Z}}\boldsymbol{Q}\to H^{2r}(S\times S',\boldsymbol{C})$. Then, using the Hodge decomposition

$$(1.1)\qquad H^{2r}(S\times S',\boldsymbol{C})\cong\bigoplus_{p+q=2r}H^{p,q}(S\times S',\boldsymbol{C})$$

of the complex cohomology, we know

$$H^{2r}(S\times S',\boldsymbol{Q})_{\mathrm{alg}}\subseteqq H^{r,r}(S\times S',\boldsymbol{C})\cap H^{2r}(S\times S',\boldsymbol{Q})=H^{2r}(S\times S',\boldsymbol{Q})_{\mathrm{Hodge}}.$$

We define

$$H^2(S,\boldsymbol{C})_{\mathrm{trans}}=\varinjlim_{U\subset S,\,\mathrm{open}}H^2(U,\boldsymbol{C}),$$

and we have the "transcendence-decomposition" of $H^2(S,\boldsymbol{C})$ with respect to the intersection numbers,

$$(1.2)\qquad H^2(S,\boldsymbol{C})\cong H^2(S,\boldsymbol{C})_{\mathrm{alg}}\oplus H^2(S,\boldsymbol{C})_{\mathrm{trans}}$$

where $H^2(S,\boldsymbol{C})_{\mathrm{alg}}=H^2(S,\boldsymbol{Q})_{\mathrm{alg}}\otimes_{\boldsymbol{Q}}\boldsymbol{C}$ (cf. Hodge and Atiyah [3], Grothendieck [1]).

Using (1.1), (1.2) and the Künneth decomposition, we make the following

Definition (1.3). The Hodge-Künneth-Transcendence-part (HKT-part) of $H^4(S\times S',\boldsymbol{C})$ is its subspace

$$H^4_{\mathrm{hkt}}(S,S')\cong\{H^{2,0}(S,\boldsymbol{C})\otimes H^{0,2}(S',\boldsymbol{C})\}\oplus\{H^{0,2}(S,\boldsymbol{C})\otimes H^{2,0}(S',\boldsymbol{C})\}\oplus\{H^{1,1}(S,\boldsymbol{C})_{\mathrm{trans}}\otimes H^{1,1}(S',\boldsymbol{C})_{\mathrm{trans}}\},$$

where $H^{1,1}(S,\boldsymbol{C})_{\mathrm{trans}}=H^{1,1}(S,\boldsymbol{C})\cap H^2(S,\boldsymbol{C})_{\mathrm{trans}}$. We let $p: H^4(S\times S',\boldsymbol{C})\to H^4_{\mathrm{hkt}}(S,S')$ denote the projection, and let $H^4_{\mathrm{hkt}}(S,S')_{\mathrm{alg}}=H^4_{\mathrm{hkt}}(S,S')\cap H^4(S\times S',\boldsymbol{Q})_{\mathrm{alg}}$.

Note that $H^4_{\text{hkt}}(S, S')$ is equal to

$$\{H^2(S, \boldsymbol{C})_{\text{trans}} \otimes H^2(S', \boldsymbol{C})_{\text{trans}}\} \cap H^{2,2}(S \times S', \boldsymbol{C}).$$

By a result of Lieberman [7], the Künneth components of an algebraic cycle class on $S \times S'$ are again algebraic and

$$H^4(S \times S', \boldsymbol{Q})_{\text{alg}} \cong \bigoplus_{p+q=4} \{H^p(S, \boldsymbol{Q}) \otimes H^q(S', \boldsymbol{Q})\}_{\text{alg}}.$$

Thus we can show the following

Lemma (1.4). *If the irregularities* $q(S)=q(S')=0$, *where* $q(S) = \dim_{\boldsymbol{C}} H^{0,1}(S, \boldsymbol{C})$, *then we have*

$$H^4(S \times S', \boldsymbol{Q})_{\text{alg}} \cong \{H^4(S, \boldsymbol{Q}) \otimes H^0(S', \boldsymbol{Q})\} \oplus \{H^0(S, \boldsymbol{Q}) \otimes H^4(S', \boldsymbol{Q})\}$$
$$\oplus \{H^2(S, \boldsymbol{Q})_{\text{alg}} \otimes H^2(S', \boldsymbol{Q})_{\text{alg}}\} \oplus H^4_{\text{hkt}}(S, S')_{\text{alg}}.$$

§ 2. Some basic properties. Throughout this section, S and S' denote non-singular projective surfaces with $q(S)=q(S')=0$.

Definition (2.1). Let X be a prime 2-cycle on $S \times S'$, and let π_i $(i=1, 2)$ be the projection of $S \times S'$ on S, S'. The prime cycle X is *degenerate* if $\dim \pi_1(X)$ or $\dim \pi_2(X)$ is less than two. We denote by $FC^2(S, S')$ $(\subseteqq C^2(S \times S'))$ the free abelian group generated by degenerate prime cycle classes, and denote by $FH^4(S, S')$ the image of $FC^2(S, S') \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$ by the cycle map cl. (Hence $FH^4(S, S') \subseteqq H^4(S \times S', \boldsymbol{Q})_{\text{alg}}$.)

Definition (2.2). Denote by $DC^2(S \times S')$ $(\subseteqq C^2(S \times S'))$ the free abelian group generated by intersections of two divisor classes on $S \times S'$, and $DH^4(S \times S') = cl\,(DC^2(S \times S') \otimes_{\boldsymbol{Z}} \boldsymbol{Q})$ $(\subseteqq H^4(S \times S', \boldsymbol{Q})_{\text{alg}})$.

Then we have

Theorem (2.3). i) $FH^4(S, S') = \{H^4(S, \boldsymbol{Q}) \otimes H^0(S', \boldsymbol{Q})\}$
$\oplus \{H^0(S, \boldsymbol{Q}) \otimes H^4(S', \boldsymbol{Q})\} \oplus \{H^2(S, \boldsymbol{Q})_{\text{alg}} \otimes H^2(S', \boldsymbol{Q})_{\text{alg}}\}$,

ii) $DH^4(S \times S') \subseteqq FH^4(S, S')$.

In particular, $p(DH^4(S \times S'))=0$. (For the map p, see (1.3).)

In fact, by the Poincaré duality, we have a natural bijection

$$\operatorname{Hom}_{\boldsymbol{C}}(H^2(S, \boldsymbol{C})_{\text{trans}}, H^2(S', \boldsymbol{C})_{\text{trans}}) \rightleftarrows H^2(S, \boldsymbol{C})_{\text{trans}} \otimes H^2(S', \boldsymbol{C})_{\text{trans}}.$$

If $X \in FC^2(S, S')$, then by the definition of $H^2(S, \boldsymbol{C})_{\text{trans}}$, the correspondence

$$X(\ \ ) : H^2(S, \boldsymbol{C})_{\text{trans}} \to H^2(S', \boldsymbol{C})_{\text{trans}}\,;\ u \mapsto X(u) = \pi_{2*}(X \cdot \pi_1^* u)$$

is zero map. i) follows from this. By taking account of the divisorial correspondences between S and S' [5], [11], ii) follows from the facts $q(S)=q(S')=0$. The last assertion follows from (1.4).

Corollary (2.4). *Let* $X \in C^2(S \times S')$ *with* $p(cl\,(X)) \neq 0$, *then* X *is not homologous to a sum of intersections of divisors.*

Corollary (2.5). *Let* $p_g(S') \geqq 1$, *where* $p_g(S') = \dim_{\boldsymbol{C}} H^{2,0}(S', \boldsymbol{C})$, *and let* $f : S \to S'$ *be a surjective morphism, then the graph* Γ_f *of* f *is not homologous to a sum of intersections of divisors.*

((2.5) follows from considering the homomorphism

$$f^* : H^{2,0}(S', \boldsymbol{C}) \to H^{2,0}(S, \boldsymbol{C}).)$$

Next we make

Definitition (2.6). By a correspondence group between S and S', we mean

$$\mathrm{Cor}^2(S, S') = C^2(S \times S')/FC^2(S, S').$$

This is considered as a generalization of the correspondence group of curves (cf. Weil [11]). The following proposition shows that HKT-part is useful for investigation of $\mathrm{Cor}^2(S, S')$.

Proposition (2.7). *There is a surjective homomorphism*

$$\overline{cl} : \mathrm{Cor}^2(S, S') \otimes_{\boldsymbol{Z}} \boldsymbol{Q} \rightarrow H^4_{\mathrm{hkt}}(S, S')_{\mathrm{alg}}$$

where $\overline{cl}$ induced by the cycle map cl.

In fact, we have the following exact commutative diagram:

$$\begin{array}{ccccccccc}
0 \longrightarrow & FC^2(S,S')\otimes_{\boldsymbol{Z}}\boldsymbol{Q} & \longrightarrow & C^2(S\times S')\otimes_{\boldsymbol{Z}}\boldsymbol{Q} & \longrightarrow & \mathrm{Cor}^2(S,S')\otimes_{\boldsymbol{Z}}\boldsymbol{Q} & \longrightarrow 0 \\
 & \downarrow{\scriptstyle cl'} & & \downarrow{\scriptstyle cl} & & \downarrow{\scriptstyle \overline{cl}} & \\
0 \longrightarrow & FH^4(S,S') & \longrightarrow & H^4(S\times S', \boldsymbol{Q})_{\mathrm{alg}} & \xrightarrow[p']{} & H^4_{\mathrm{hkt}}(S,S')_{\mathrm{alg}} & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 &
\end{array}$$

where p', cl' are the restrictions of p, cl, respectively. (We note that taking some adequate equivalence relation $\sim$ finer than homological equivalence, instead of rational equivalence, we have the correspondences $\mathrm{Cor}^2_{\sim}(S, S')$, and surjective homomorphism $\overline{cl}_{\sim} : \mathrm{Cor}^2_{\sim}(S, S') \otimes_{\boldsymbol{Z}} \boldsymbol{Q} \rightarrow H^4_{\mathrm{hkt}}(S, S')_{\mathrm{alg}}$. For example, for homological equivalence we have the isomorphism $\overline{cl}_{\mathrm{hom}} : \mathrm{Cor}^2_{\mathrm{hom}}(S, S') \otimes_{\boldsymbol{Z}} \boldsymbol{Q} \rightleftarrows H^4_{\mathrm{hkt}}(S, S')_{\mathrm{alg}}$.)

For the remainder of this note, we investigate the HKT-parts of algebraic 2-cycles on products of certain two surfaces.

§ 3. Singular $K3$ surfaces. By a singular $K3$ surface S, we mean an algebraic $K3$ surface (defined over $\boldsymbol{C}$) whose Picard number $\rho(S)$ equals to $\dim_{\boldsymbol{C}} H^{1,1}(S, \boldsymbol{C})$. (Here we let $\rho(S) = \dim_{\boldsymbol{C}} H^2(S, \boldsymbol{C})_{\mathrm{alg}}$.) We note that a singular $K3$ surface S satisfies $q(S)=0$, $p_g(S)=1$ and $H^{1,1}(S, \boldsymbol{C})_{\mathrm{trans}} \cong 0$.

We assume that S and S' are singular $K3$ surfaces. For the details on these surfaces, see Shioda and Inose [9]. Let ω and ω' be respectively bases of $H^0(S, \Omega^2_S)$ and $H^0(S', \Omega^2_{S'})$, and let $\{\gamma_1, \gamma_2\}$ and $\{\gamma'_1, \gamma'_2\}$ be respectively bases of $H_2(S, \boldsymbol{Q})_{\mathrm{trans}}$ and $H_2(S', \boldsymbol{Q})_{\mathrm{trans}}$. Let

$$\tau = \int_{\gamma_1} \omega \Big/ \int_{\gamma_2} \omega \quad \text{and} \quad \eta = \int_{\gamma'_1} \omega' \Big/ \int_{\gamma'_2} \omega'.$$

Let E_τ denote the elliptic curve of the form $\boldsymbol{C}/\boldsymbol{Z} + \tau\boldsymbol{Z}$. Then we have

Theorem (3.1). $H^4_{\mathrm{hkt}}(S, S')_{\mathrm{Hodge}} \cong \mathrm{Cor}(E_\tau, E_\eta) \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$

$$\cong \mathrm{Hom}(E_\tau, E_\eta) \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$$

where $H^4_{\mathrm{hkt}}(S, S')_{\mathrm{Hodge}} = H^4_{\mathrm{hkt}}(S, S') \cap H^4(S \times S', \boldsymbol{Q})$ *and* $\mathrm{Cor}(E_\tau, E_\eta)$ *denotes the correspondence group between* E_τ *and* E_η (cf. Weil [11]).

§ 4. Some quotient surfaces. Let C_i be the algebraic curve in $\boldsymbol{P}^2$ defined by

$$u_2^n = \prod_{j=1}^n (u_1 - a_{ij} u_0) \quad (i = 1, 2, 3, 4, \text{ and } n : \text{prime number}, > 2).$$

(We note that if $a_{ij}=\zeta^j$, $\zeta=\exp(2\pi\sqrt{-1}/n)$, for all $j=1,\cdots,n$, then C_i is the Fermat curve of degree n.) Let G_n denote the group of n-th roots of unity: $G_n=\langle\zeta\rangle$. We introduce an action of G_n on C_i:

$$(u_0:u_1:u_2)\mapsto(u_0:u_1:\zeta u_2).$$

We define an embedding $i_r: G_n\to G_n\times G_n$ $(1\leqq r\leqq n-1)$ by $i_r(\zeta)=(\zeta,\zeta^r)$, and we set $G^{(r)}=\mathrm{Im}(i_r)$. Then $G^{(r)}$ and $G^{(s)}$ act naturally on $\tilde{S}=C_1\times C_2$ and $\tilde{S}'=C_3\times C_4$, and $G^{(r,s)}=G^{(r)}\times G^{(s)}$ acts on $\tilde{S}\times\tilde{S}'$ $(1\leqq r, s\leqq n-1)$. Let S_r and S'_s be non-singular models of $\tilde{S}/G^{(r)}$ and $\tilde{S}'/G^{(s)}$ respectively $(1\leqq r, s\leqq n-1)$. Note that one can take S_1 (resp. S'_1) to be the surface in $\boldsymbol{P}^3$ defined by

$$\prod_{j=1}^n(x_3-a_{1j}x_2)=\prod_{j=1}^n(x_1-a_{2j}x_0)$$

$$(\text{resp. } \prod_{j=1}^n(x_3-a_{3j}x_2)=\prod_{j=1}^n(x_1-a_{4j}x_0)) \quad (\text{cf. Sasakura [8]}).$$

(We also note $q(S_1)=q(S'_1)=0$ and that if C_i are the Fermat curves, for all i, then S_1 and S'_1 are the Fermat surfaces.)

By a simple calculation, we have

$$(4.1)\qquad H^4_{\mathrm{hkt}}(\tilde{S},\tilde{S}')_{\mathrm{alg}}\cong\bigoplus_{1\leqq r,s\leqq n-1}\{H^4_{\mathrm{hkt}}(\tilde{S},\tilde{S}')_{\mathrm{alg}}\}^{G^{(r,s)}},$$

where the right side is $G^{(r,s)}$-invariant part, and we have a natural homomorphism

$$(4.2)\qquad \{H^4_{\mathrm{hkt}}(\tilde{S},\tilde{S}')_{\mathrm{alg}}\}^{G^{(r,s)}}\to H^4_{\mathrm{hkt}}(S_r,S'_s)_{\mathrm{alg}}\qquad(1\leqq r,\ s\leqq n-1).$$

Since $H^{2,0}(\tilde{S},\boldsymbol{C})^{G^{(r)}}\rightrightarrows H^{2,0}(S_r,\boldsymbol{C})$, the above homomorphism is non-zero.

Now we let $J(C_i)$ be the Jacobian variety of C_i $(i=1,2,3,4)$ and $J=\mathrm{Hom}(J(C_1),J(C_3))\otimes\mathrm{Hom}(J(C_2),J(C_4))$. Then, from (4.1) and (4.2), we have a natural homomorphism

$$\theta^{r,s}: J\to H^4_{\mathrm{hkt}}(S_r,S'_s)_{\mathrm{alg}}\qquad(1\leqq r,\ s\leqq n-1).$$

The following facts are also checked easily, by using (4.1) and (4.2).

Theorem (4.3). *There exists (r,s), $1\leqq r, s\leqq n-1$, such that* $\mathrm{Im}(\theta^{r,s})\neq 0$.

Theorem (4.4). *For isogenies $u: J(C_1)\to J(C_3)$ and $v: J(C_2)\to J(C_4)$ we have $\theta^{r,s}(u\otimes v)\neq 0$ for all $1\leqq r, s\leqq n-1$.*

Thus for the isogenies u and v, $\theta^{1,1}(u\otimes v)$ is the HKT-part of an algebraic cycle class on $S_1\times S'_1$ which is not a sum of intersection of divisors. (More detailed structures of the HKT-part of the product of the quotient surfaces $S_1\times S'_1$ will be given elsewhere.)

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