69. *On the Solvability of Goursat Problems and a Function of Number Theory*

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1. Introduction. In this paper we shall study the reduced Goursat problem with constant coefficients:

(1.1) $$Lu=(a\partial_1^{-1}\partial_2+\varepsilon+b\partial_1\partial_2^{-1}+c\partial_1^2\partial_2^{-2})u=h(x)$$

where $x=(x_1,x_2)\in C^2$, $\partial_i=\partial/\partial x_i$ $(i=1,2)$ and ∂_i^{-1} is the integration with respect to the variable x_i from the origin to x_i.

If the roots $\lambda_1,\lambda_2,\lambda_3$ of the characteristic equation of L;

(1.2) $$a\lambda^3+\varepsilon\lambda^2+b\lambda+c=0$$

satisfy the "Alinhac-Leray condition" $|\lambda_1|\leqq|\lambda_2|<|\lambda_3|$ the solvability and the uniqueness of (1.1) are proved by S. Alinhac in [1] under some additional conditions. Whereas, if the condition is not satisfied few results are known. The best work known is that of Leray's for (1.1) with $c=0$ in [2]. He introduced the number-theoretical function $\rho(\theta)$ (cf. [2]) and expressed a sufficient condition for the solvability and uniqueness of (1.1) for $c=0$ in terms of $\rho(\theta)$.

The purpose of this paper is to study the case $c\neq 0$ without assuming the Alinhac-Leray condition. We introduce a function $\rho(\theta_1,\theta_2)$ as a natural extension of the Leray's auxiliary function $\rho(\theta)$ which describes the transcendency of θ_1 and θ_2. In terms of this function we shall characterize the range of the operator L. As a result we reveal a close connection between the algebraic-transcendental properties of the characteristic roots and the solvability and uniqueness. We remark that the results here can be extended to a wider class of equations with multiple characteristic roots.

2. Statement of theorems. Without loss of generality we may assume that $ac\neq 0$. Moreover, by the linear change of variables such as $rx_1=z_1$, $x_2=z_2$ $(r\neq 0)$ we may assume that eq. (1.2) has the root 1 and that the absolute values of other roots do not exceed 1. Since we are interested in the case where the Alinhac-Leray condition is not satisfied we assume $0<|\lambda_1|\leqq|\lambda_2|=1$. Let H_0 be the set of functions analytic at the origin. Then

Theorem 2.1. *If the roots* λ_1,λ_2, 1 *of eq.* (1.2) *are not distinct the map* $L:H_0\to H_0$ *is bijective.*

In view of this theorem we shall consider the case where the roots $\lambda_1,\lambda_2,1$ are distinct. Let I_k be defined by

$$I_k=\lambda_1\lambda_2\{\lambda_1^{k+2}(1-\lambda_2)+\lambda_2^{k+2}(\lambda_1-1)+\lambda_2-\lambda_1\}.$$

Then we have

Proposition 2.1. *The map $L: H_0 \to H_0$ is injective iff I_k does not vanish for $k=1,2,\cdots$.*

To study the range of L we consider the following three cases; A) $|\lambda_1|=|\lambda_2|=1$, B) $|\lambda_1-1|=|\lambda_1-\lambda_2|$ and $|\lambda_1|<1$, C) otherwise.

Case A) Write $\lambda_j=\exp(2\pi i\theta_j)$, $0\leqq\theta_j<1$ $(j=1,2)$ and define the function $\rho(\theta_1,\theta_2)$ by

$$\rho(\theta_1,\theta_2)=\liminf_{k\to\infty}\inf_{p,q\in Z}(|k\theta_1-p|^{1/k}+|k\theta_2-q|^{1/k}).$$

Note that the function $\rho(\theta,0)$ is the one introduced by J. Leray in [2]. Then we have

Theorem 2.2. *Let (λ_1,λ_2) be in Case* A). *Then $LH_0=H_0$ if and only if $\rho(\theta_1,\theta_2)>0$.*

Remarks. a) It follows from the definition of I_k that I_k vanishes iff both θ_1 and θ_2 are rational. Hence, by Theorem 2.1 L is bijective iff $\rho(\theta_1,\theta_2)>0$.

b) If we define Δ as the set of all real θ satisfying $\rho(\theta,0)=0$ we can see that $m_1(\Delta)=0$ and that the set Δ has the density of continuum (cf. [3]). Then the set of all (θ_1,θ_2) such that $\rho(\theta_1,\theta_2)=0$ are contained in $\Delta\times\Delta$ and contains all the points $(l\theta,m\theta)$ where $\theta\in\Delta$ and l and m are integers.

Next we shall study the case $\rho(\theta_1,\theta_2)=0$. First we consider the case where both θ_1 and θ_2 are rational. We determine the integers s_1 and s_2 by $\theta_1=r_1/s_1$, $\theta_2=r_2/s_2$ where r_1,s_1 and r_2,s_2 are relatively prime non-negative integers respectively. We denote the least common multiple of s_1 and s_2 by s_0. Then

Theorem 2.3. *A function $h(x)\in H_0$ is in the image LH_0 of H_0 by L iff $h(x)$ satisfies, for all $k=s_0p-1$, s_0p-2 $(p=1,2,\cdots)$,*

$$\sum_{j=1}^{k}h_{k-j,j-1}I_{k-j}=0 \tag{2.1}$$

where $h(x)=\sum h_{p,q}x_1^p x_2^q/(p!\,q!)$. The kernel of the map $L: H_0\to H_0$ is an infinite-dimensional vector space.

To study the case where either θ_1 or θ_2 is irrational we need some preparations.

For each $\eta\geqq 0$ we define the class of entire functions B_η by

$$B_\eta=\{h\in H_0\,;\,|h_\alpha|\leqq M_0 r_1^{|\alpha|}(\alpha_1!\,\alpha_2!)^{1-\eta}\text{ for some }M_0,r_1>0\}.$$

Here $h(x)=\sum h_\alpha x^\alpha/\alpha!$. Note that $B_0=H_0$. Let $t=[a_1,a_2,\cdots]$ be a continued fraction expansion of irrational number t $(0<t<1)$ with

$$a_1=[1/t],\quad \alpha_2=1/t-a_1,\ \cdots,\ a_n=[1/\alpha_n],\quad \alpha_{n+1}=1/\alpha_n-a_n,$$

where $[\mu]$ denotes the largest integer $\leqq\mu$. Then we determine the integer q_n $(n=1,2,\cdots)$ by the relation $q_n=a_nq_{n-1}+q_{n-2}$, $q_{-1}=0$, $q_0=1$ $(n=1,2,\cdots)$ and set, for $\gamma\geqq 0$,

$J_\gamma=\{t\,;\,0<t<1,\ t$ is irrational and satisfies $(a_{n+1})^{1/q_n}=O(q_n^\gamma)$ as $n\to\infty\}$. Here if $\gamma=0$ we understand that $O(q_n^\gamma)=O(1)$. We easily see that $J_{\gamma'} \subsetneqq J_\gamma$ for every $0\leqq\gamma'<\gamma$ and that J_γ has the density of continuum. Moreover we can prove that $\rho(\theta,0)=0$ for every $\theta\in J_\gamma\backslash J_0$ $(\gamma>0)$. Note that $\rho(l\theta, m\theta)=0$ for every $\theta\in J_\gamma\backslash J_0$ and every integers l and m. Then we have

Theorem 2.4. *The map $L: H_0\to H_0$ is injective and the image $L\,H_0$ has the following properties:*

a) *Suppose that θ_1 or θ_2 is in J_γ for some $\gamma>0$. Then $L\,H_0$ contains B_η for every $\eta\geqq\gamma$.*

b) *Let m_j $(j=1,2)$ be arbitrary positive integers and let $m_0=\min(m_1,m_2)$. If $\theta_j=m_j\theta-[m_j\theta]$ $(j=1,2)$ for some $\theta\in J_\gamma\backslash J_{\gamma'}$ $(\gamma'<\gamma)$ we have*

$$L\,H_0\supsetneqq B_\eta \text{ for all } \eta\geqq m_0\gamma, \qquad L\,H_0\not\supset B_\eta \text{ for all } 0\leqq\eta<\gamma'.$$

It follows from b) and the definition of J_γ that for an arbitrary $\gamma>0$ there exists a set $\Omega_\gamma\subset R^2$ with the density of continuum such that, for every $(\theta_1,\theta_2)\in\Omega_\gamma$, $L\,H_0\supsetneqq B_\eta$ if $\eta\geqq\gamma$ and $L\,H_0\not\supset B_\eta$ if $0\leqq\eta<\gamma$.

Case B) We set $\lambda_1=r\exp(\pi i\theta)$, $\lambda_2=\exp(2\pi i\theta)$ where $-1<r<1$. Then

Theorem 2.5. *For every r $(-1<r<1)$ there exists a set F of real numbers with $m_1(F)=0$ such that if θ is not in F the map $L: H_0\to H_0$ is bijective. Similarly, for every real number θ there exists a set $\tilde F\subset(-1,1)$ with $m_1(\tilde F)=0$ such that if r is not in $\tilde F$ the map L is bijective. Here $m_1(\cdot)$ denotes the Lebesgue measure in R^1.*

Case C) Let (λ_1,λ_2) be in Case C). Then

Theorem 2.6. *Suppose that $I_k\neq 0$ for $k=1,2,\cdots$. Then the map $L: H_0\to H_0$ is bijective. While if I_k vanishes exactly for $k=k_1,\cdots,k_l$, a function $h\in H_0$ is in the image $L\,H_0$ iff $h(x)$ satisfies (2.1) for $k=k_1,\cdots,k_l$. Furthermore the kernel of L is a finite-dimensional non-trivial vector space.*

References

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