22. *Remarks on the Deficiencies of Algebroid Functions of Finite Order*

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1. Introduction. Edrei and Fuchs [1] established the following interesting theorem:

Theorem A. *Let $f(z)$ be a meromorphic function of order λ, $0<\lambda<1$. Put*

$$u=1-\delta(0,f) \quad \text{and} \quad v=1-\delta(\infty,f), \quad 0\leqq u, v\leqq 1,$$

where $\delta(a,f)$ denotes the Nevanlinna deficiency of a value a. Then we have

$$u^2+v^2-2uv\cos\pi\lambda\geqq\sin^2(\pi\lambda).$$

Further, if $u<\cos\pi\lambda$, then $v=1$; if $v<\cos\pi\lambda$, then $u=1$.

This beautiful and elegant theorem solves completely the problem of finding relations between any two deficiencies of a meromorphic function of order less than one. A little later, Edrei [2] showed that the order λ in the theorem may be replaced by the lower order μ.

Shea [4] obtained a result which concerns with the Valiron deficiency $\Delta(a,f)$ instead of $\delta(a,f)$. That is, he proved

Theorem B. *Let $f(z)$ be a meromorphic function of order λ, $0<\lambda<1$, whose zeros lie on the negative real axis, and whose poles lie on the positive real axis. Put*

$$X=1-\Delta(0,f) \quad \text{and} \quad Y=1-\Delta(\infty,f).$$

Then, when $1/2\leqq\lambda<1$, we have

$$X^2+Y^2-2XY\cos\pi\lambda\leqq\sin^2(\pi\lambda).$$

When $0<\lambda<1/2$, the above inequality still holds provided

$$X\geqq\cos(\pi\lambda) \quad \text{and} \quad Y\geqq\cos(\pi\lambda).$$

The purpose of this paper is to extend these theorems to n-valued algebroid functions of order less than one. Our results are as follows:

Theorem 1. *Let $f(z)$ be an n-valued algebroid function of order λ, $0<\lambda<1$, defined by the irreducible equation*

$$A_0(z)f^n+A_1(z)f^{n-1}+\cdots+A_n(z)=0, \tag{1.1}$$

where $A_0(z), A_1(z), \cdots, A_n(z)$ are entire functions without common zeros, and we suppose that 0 is not a Valiron deficient value for $A_0(z)$.

Let a_j, $j=1,\cdots,n$, be mutually distinct values, and put

$$u_j=1-\delta(a_j,f) \quad \text{and} \quad v=1-\delta(\infty,f), \quad 0\leqq u_j, v\leqq 1. \tag{1.2}$$

Then, there is at least one a_ν, $1\leqq\nu\leqq n$, such that

$$u_\nu^2+v^2-2u_\nu v\cos\pi\lambda\geqq n^{-2}\sin^2(\pi\lambda). \tag{1.3}$$

If $u_\nu < n^{-1}\cos\pi\lambda$, *then* $v \geqq 1/n$; *if* $v < n^{-1}\cos\pi\lambda$, *then* $u_\nu \geqq 1/n$.

Theorem 2. *Let* $f(z)$ *be an* n*-valued algebroid function of order* λ, $0<\lambda<1$, *whose poles lie on the positive real axis. Let* a_j, $j=1,\cdots,n$, *be mutually distinct values and put*

(1.4) $$X_j = 1-\Delta(a_j, f) \quad \text{and} \quad Y=1-\Delta(\infty, f).$$

We suppose that zeros of $f(z)-a_j$, $j=1,\cdots,n$, *lie wholly on the negative real axis. Then, when* $1/2\leqq\lambda<1$, *we have*

(1.5) $$X_j^2+Y^2-2X_jY\cos\pi\lambda \leqq \sin^2(\pi\lambda), \qquad j=1,\cdots,n.$$

When $0<\lambda<1/2$, *the same inequality still holds for some pair* (X_ν, Y), *provided*

(1.6) $$X_\nu \geqq \cos\pi\lambda \quad \text{and} \quad Y\geqq\cos\pi\lambda.$$

2. Preliminaries. Let $f(z)$ and a_j, $j=1,\cdots,n$, be as in Theorem 1. Let $Y_j(z)$ be the j-th determination of $f(z)$, $1\leqq j\leqq n$. Put

$$A(z)=\max(1, |A_0(z)|, \cdots, |A_n(z)|),$$
$$g(z)=\max(1, |g_1(z)|, \cdots, |g_n(z)|),$$

in which $g_j(z)=A_0(z)a_j^n+A_1(z)a_j^{n-1}+\cdots+A_n(z)$, and

$$\mu(r, A)=\frac{1}{2\pi n}\int_0^{2\pi}\log A(re^{i\theta})d\theta.$$

Then, by a theorem of Valiron [6], we have

(2.1) $$|\mu(r, A)-T(r, f)|=O(1).$$

Ozawa [3] showed that

(2.2) $$\mu(r, g)=\mu(r, A)+O(1),$$

(2.3) $$\sum_{j=1}^{n}\log^+|y_j(z)|\leqq\log\left|\frac{A(z)}{A_0(z)}\right|+O(1).$$

We put $f_j(z)=g_j(z)/A_0(z)$. Then, by [5, p. 2, Prop. 4 (ii)], we have (using (2.1))

(2.4) $$T(r, f_j)<n\mu(r, A)+O(1)\leqq nT(r, f)+O(1),$$

from which we see that $f_j(z)$ are meromorphic functions of order at most λ.

Then, we have

(2.5) $$\begin{aligned}\sum_{j=1}^{n}T(r, f_j) &\geqq \sum_{j=1}^{n}\frac{1}{2\pi}\int_0^{2\pi}\log^+\left|\frac{g_j(re^{i\theta})}{A_0(re^{i\theta})}\right|d\theta+N\left(r, \frac{1}{A_0}\right)\\
&\geqq\frac{1}{2\pi}\int_0^{2\pi}\log^+(\max_j|g_j(re^{i\theta})|)d\theta\\
&\quad -\frac{1}{2\pi}\int_0^{2\pi}\log^+|A_0(re^{i\theta})|d\theta+N\left(r, \frac{1}{A_0}\right)\\
&=\frac{1}{2\pi}\int_0^{2\pi}\log g(re^{i\theta})d\theta-m(r, A_0)+N\left(r, \frac{1}{A_0}\right)\\
&=n\mu(r, g)-m(r, A_0)+N(r, 1/A_0)\\
&=n\mu(r, g)-m(r, 1/A_0)+O(1).\end{aligned}$$

Since 0 is not a Valiron deficient value for $A_0(z)$, we have

(2.6) $$m(r, 1/A_0)=o(T(r, A_0))=o(\mu(r, A))=o(T(r, f)).$$

By (2.1), (2.2), (2.5) and (2.6), we obtain

$$(2.7)\qquad nT(r, f)<n\mu(r, A)+O(1)\leqq\sum_{j=1}^{n} T(r, f_j)+o(T(r, f)).$$

3. **Proof of Theorem 1.** We make use of the techniques of Edrei-Fuchs [1]. Since each $f_j(z)=g_j(z)/A_0(z)$ is a function of order $\leqq\lambda<1$, we obtain as in [1, p. 239],

$$(3.1)\qquad T(r, f_j)\leqq\int_0^\infty N_j(t, 0)P(t, r, \beta_j)dt+\int_0^\infty N(t, \infty)P(t, r, \pi-\beta_j)dt,$$

where

$$P(t, r, \gamma)=\pi^{-1}r\sin\gamma/(t^2+2tr\cos\gamma+r^2)\qquad(0<\gamma<\pi)$$

and $\beta_j=\beta_j(r)$ is a number such that $0<\beta_j<\pi$. $N_j(t, 0)$ and $N(t, \infty)$ denote the counting functions of $1/f_j(z)$ and $A_0(z)$, respectively.

By (3.1) and (2.7), we get

$$nT(r, f)\leqq\sum_{j=1}^{n}\int_0^\infty nN(t; a_j, f)P(t, r, \beta_j)dt$$
$$+\sum_{j=1}^{n}\int_0^\infty nN(t; \infty, f)P(t, r, \pi-\beta_j)dt.$$

Let U_j and V be such that $U_j>u_j$ and $V>v$. Then, by the definition of deficiency

$$N(t; a_j, f)<U_jT(t, f)\quad\text{and}\quad N(t; \infty, f)<VT(t, f)\quad(t\geqq t_0).$$

As in [1, p. 240], we make use of the notion of Pólya peaks $\{r_m\}$. Then we deduce

$$(3.2)\qquad (1+o(1))T(r_m, f)\leqq\sum_{j=1}^{n} U_jT(r_m, f)\left\{\int_0^{r_m}\left(\frac{t}{r_m}\right)^{\lambda-\varepsilon}P(t, r_m, \beta_j)dt\right.$$
$$\left.+\int_{r_m}^\infty\left(\frac{t}{r_m}\right)^{\lambda+\varepsilon}P(t, r_m, \beta_j)dt\right\}$$
$$+\sum_{j=1}^{n} VT(r_m, f)\left\{\int_0^{r_m}\left(\frac{t}{r_m}\right)^{\lambda-\varepsilon}P(t, r_m, \pi-\beta_j)dt\right.$$
$$\left.+\int_{r_m}^\infty\left(\frac{t}{r_m}\right)^{\lambda+\varepsilon}P(t, r_m, \pi-\beta_j)dt\right\}+\eta(r_m),$$

where $\eta(r_m)$ is a quantity such that $\eta(r_m)=O(1/r)$.

Writing $t=sr_m$, we obtain

$$\int_0^{r_m}\left(\frac{t}{r_m}\right)^{\lambda-\varepsilon}P(t, r_m, \beta_j)dt+\int_{r_m}^\infty\left(\frac{t}{r_m}\right)^{\lambda+\varepsilon}P(t, r_m, \beta_j)dt$$
$$=\int_0^\infty s^{\lambda+\varepsilon}P(s, 1, \beta_j)dt+\int_0^1(s^{\lambda-\varepsilon}-s^{\lambda+\varepsilon})P(s, 1, \beta_j)ds=\frac{\sin\beta_j\lambda}{\sin\pi\lambda}+\tau,$$

where $0<s^{\lambda-\varepsilon}-s^{\lambda+\varepsilon}<\tau$ $(0\leqq s\leqq 1)$. Let $r_m\to\infty$ and then make (in this order) the transition to the limit $\varepsilon\to 0$, $\tau\to 0$, $U_j\to u$, $V\to v$. We argue similarly for the terms including $\pi-\beta_j$. Thus we get

$$(3.3)\qquad 1\leqq\sum_{j=1}^{n}\max_{0\leqq\beta_j\leqq\pi}\{(u_j\sin\beta_j\lambda+v\sin(\pi-\beta_j)\lambda)/\sin\pi\lambda\}.$$

Since $u_j\sin\beta_j\lambda+v\sin(\pi-\beta_j)\lambda$ is a continuous function of β_j, we can find γ_j for which the $\max_{0\leqq\beta_j\leqq\pi}$ in (3.3) is attained. Hence

(3.4) $\sin \pi\lambda \leqq \sum_{j=1}^{n} \{u_j \sin \gamma_j\lambda + v \sin(\pi-\gamma_j)\lambda\} \leqq n\{u_\nu \sin \gamma_\nu\lambda + v \sin(\pi-\gamma_\nu)\lambda\}$

for a ν, $1 \leqq \nu \leqq n$. Thus

(3.5) $$\sin^2\pi\lambda \leqq n^2\{u_\nu \sin \gamma_\nu\lambda - v \sin \gamma_\nu\lambda \cos \pi\lambda + v \sin \pi\lambda \cos \gamma_\nu\lambda\}^2$$
$$\leqq n^2\{(u_\nu - v\cos\pi\lambda)^2 + v^2 \sin^2\pi\lambda\} = n^2\{u_\nu^2 + v^2 - 2u_\nu v \cos \pi\lambda\},$$

which proves the inequality in Theorem 1.

If $v < n^{-1}\cos\pi\lambda$, then from (3.5) we see that $\gamma_\nu \neq 0$, and by (3.4)

$$nu_\nu \sin\gamma_\nu\lambda \geqq \sin\pi\lambda - \sin(\pi-\gamma_\nu)\lambda\cos\pi\lambda$$
$$\geqq \sin\pi\lambda\cos(\pi-\gamma_\nu)\lambda - \sin(\pi-\gamma_\nu)\lambda\cos\pi\lambda$$
$$= \sin\gamma_\nu\lambda, \quad u_\nu \geqq 1/n.$$

The case $u_\nu < n^{-1}\cos\pi\lambda$ is treated similarly.

4. Proof of Theorem 2. We follow the method of Shea [4]. Applying Shea's representation to meromorphic function $f_j(z) = g_j(z)/A_0(z)$ and using (2.4), we have

(4.1) $$nT(r, f) \geqq \int_0^\infty nN(t; a_j, f)P(t, r, \beta_j)dt + \int_0^\infty nN(t; \infty, f)P(t, r, \pi-\beta_j)dt - A\log r$$

with a suitable constant $A > 0$ (see [4, p. 215]).

Let $\overline{X}_j$, $\overline{Y}$ be such that $0 < \overline{X}_j < X_j$ and $0 < \overline{Y} < Y$. Then

(4.2) $N(t; a_j, f) \geqq \overline{X}_j T(t, f)$ and $N(t; \infty, f) \geqq \overline{Y}T(t, f)$ $(t \geqq t_0)$

We argue as in [4, p. 216]. Let μ be the lower order of the algebroid function $f(z)$, and choose any positive number ρ such that $\mu \leqq \rho \leqq \lambda$, and let $\{r_m\}$ be a sequence of Pólya peaks of the second kind, of order ρ, for the function $T(r, f)$. Then we obtain by (4.1)

$$T(r_m, f) \geqq \overline{X}_j T(r_m, f)(1+o(1)) \int_s^S (t/r_m)^\rho P(t, r_m, \beta_j)dt + \overline{Y}T(r_m, f)(1+o(1)) \int_s^S (t/r_m)^\rho P(t, r_m, \pi-\beta_j)dt - A\log r \quad (m \to \infty),$$

where s and S run over the associated sequences $\{s_m\}$ and $\{S_m\}$ for $\{r_m\}$ [4, p. 208]. Making the change of variable $s = t/r_m$ and divided by $T(r_m, f)$, we get

$$1 + o(1) \geqq \overline{X}_j \int_{s_m/r_m}^{S_m/r_m} s^\rho P(s, 1, \beta_j)ds + \overline{Y} \int_{s_m/r_m}^{S_m/r_m} s^\rho P(s, 1, \pi-\beta_j)ds \quad (m \to \infty).$$

Thus

$$1 \geqq \overline{X}_j \int_0^\infty s^\rho P(s, 1, \beta_j)ds + \overline{Y} \int_0^\infty s^\rho P(s, 1, \pi-\beta_j)ds.$$

Evaluating these integrals and letting $\overline{X}_j \to X_j$, $\overline{Y} \to Y$, we obtain

(4.3) $$\sin\pi\rho \geqq X_j \sin\beta_j\rho + Y\sin(\pi-\beta_j)\rho \quad (\mu \leqq \rho \leqq \lambda)$$

for any j, $1 \leqq j \leqq n$. (4.3) holds for any β_j, $0 < \beta_j < \pi$, but since the right hand side is continuous, (4.3) holds for $0 \leqq \beta_j \leqq \pi$.

We put $\rho = \lambda$ and $\beta_j = \lambda^{-1}\tan^{-1}((X_j - Y\cos\pi\lambda)/(Y\sin\pi\lambda))$. Then we obtain easily the inequality (1.5). We note that the supposition (1.6) insures that $0 \leqq \beta_\nu \leqq \pi$ when $\lambda < 1/2$.

References

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