16. *Cauchy Problem for Hyperbolic Differential Operators with Double Characteristic Roots*

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1. Introduction. Let us consider the Cauchy problem for hyperbolic differential operators with double characteristic roots. Already we have some sufficient conditions for this Cauchy problem to be well-posed in C^∞-class, cf. [5]. On the other hand, we also know that without such a condition, this Cauchy problem is well-posed in κ-Gevrey class, $1\le\kappa<2$, cf. [2], [3].

In this paper, we introduce a number $\kappa^*\in[2,\infty]$ which shall be determined according to a given operator and show that if $1\le\kappa<\kappa^*$, the above Cauchy problem is well-posed in κ-Gevrey class.

2. Definitions. We consider the Cauchy problem

$$\text{(C)}\quad \begin{cases} P[u]=D_t^m u+\sum_{j=0}^{m-1}\sum_{|\nu|\le m-j} a_{\nu j}D_x^\nu D_t^j u=f(x,t), & (x,t)\in\Omega \\ D_t^j u|_{t=0}=\phi_j(x), & j=0,1,\cdots,m-1 \end{cases}$$

where $\Omega=\boldsymbol{R}^n\times[0,T]$, $T>0$, $D_t=-i\dfrac{\partial}{\partial t}$, $D_j=-i\dfrac{\partial}{\partial x_j}$, $\nu=(\nu_1,\cdots,\nu_n)$; ν_j are non-negative integers, $D_x^\nu=D_1^{\nu_1}\cdots D_n^{\nu_n}$. Let $P_j(x,t;D_x,D_t)$ be the homogeneous part of degree j in (D_x,D_t) of $P(x,t;D_x,D_t)$.

We say that $a(x,t)$ belongs to a class $\gamma^{(\kappa)}$, $\phi(x)$ to $\Gamma^{(\kappa)}$ and $\psi(x,t)$ to $\Gamma^{r(\kappa)}$; $r=0,1,\cdots,\infty$, if there exist constants $\rho>0$ and $C\ge0$ according to $a(x,t)$, $\phi(x)$ and $\psi(x,t)$ respectively such that

$$|D_x^\nu D_t^j a(x,t)|\le C\frac{(j+|\nu|)!^\kappa}{\rho^{j+|\nu|}},\ (x,t)\in\Omega,\qquad \text{for any } j \text{ and } \nu,$$

$$\|D_x^\nu\phi\|\le C\frac{|\nu|!^\kappa}{\rho^{|\nu|}},\qquad \text{for any } \nu,$$

$$\|D_x^\nu D_t^j\psi(t)\|\le C\frac{(j+|\nu|)!^\kappa}{\rho^{j+|\nu|}},\ 0\le t\le T,\qquad \text{for any } j\le r \text{ and any } \nu,$$

respectively, where $\|\ \|$ denotes the L_x^2-norm.

We also say that $h(x,t,\xi)$ belongs to $\mathscr{B}_t^k[S^r(\kappa)]$ if 1) $h(x,t,\xi)$ is homogeneous of degree r in ξ, and 2) there exists a constant $\rho>0$ such that for any $j\le k$ and any α, β,

$$|D_t^j D_x^\beta D_\xi^\alpha h(x,t,\xi)|\le C_{j\alpha}\frac{|\beta|!^\kappa}{\rho^{|\beta|}},\qquad (x,t)\in\Omega,\ |\xi|=1,$$

where $C_{j\alpha}$ is a constant independent of β.

3. Result. We assume the following three conditions:

i) *The coefficients* $a_{\nu j}(x,t)$ *belong to* $\gamma^{(\kappa)}$.

ii) *The characteristic polynomial* $P_m(x,t;\xi,\tau)$ *can be decomposed as*

$$P_m(x,t;\xi,\tau)=\prod_{j=1}^{m-s}(\tau-\lambda_j(x,t,\xi))\prod_{j=1}^{s}(\tau-\mu_j(x,t,\xi)),\qquad 2s\leq m,$$

where λ_j, μ_j *are real-valued and belong to* $\mathcal{B}_t^{m-1}[\mathcal{S}^1(\kappa)]$. *Moreover* $\{\lambda_j\}_{j=1,\cdots,m-s}$ *and* $\{\mu_j\}_{j=1,\cdots,s}$ *are distinct in each group, but* μ_j *may coincide only with* λ_j *somewhere in* $\Omega\times\boldsymbol{R}^n\backslash\{0\}$, $j=1,\cdots,s$.

iii) *For each* j $(1\leq j\leq s)$, *there exists* $a_j(x,t,\xi)\in\mathcal{B}_t^0[\mathcal{S}^0(\kappa)]$ *such that* $\{\tau-\mu_j,\tau-\lambda_j\}=t^{-1}(\mu_j-\lambda_j)a_j$, *where* $\{\ ,\ \}$ *denotes the Poisson's bracket.*

Now, let $L_j(x,t,\xi)$ be Levi's functions, namely

$$L_j(x,t,\xi)=P'_{m-1}\left(x,t,\xi,\frac{\mu_j+\lambda_j}{2}\right),\qquad j=1,\cdots,s,$$

where

$$P'_{m-1}=\frac{1}{2}\partial_\tau D_t P_m+\frac{1}{2}\sum_{j=1}^{n}\partial_{\xi_j}D_{x_j}P_m-P_{m-1};\quad \partial_\tau=\frac{\partial}{\partial\tau},\quad \partial_{\xi_j}=\frac{\partial}{\partial\xi_j}.$$

We define the numbers $\{\rho_j,\sigma_j\}_{j=1,\cdots,s}$ as follows:

$$\rho_j=\inf\left\{\rho\geq 1;\ \frac{t^\rho L_j(x,t,\xi)}{\mu_j-\lambda_j}\in\mathcal{B}_t^0[\mathcal{S}^{m-2}(\kappa)]\right\}.$$

If the set of ρ in the right-hand term is empty, then $\rho_j=\infty$.

$$\sigma_j=\sup\{\sigma\geq 0;\ t^{-\sigma}L_j(x,t,\xi)\in\mathcal{B}_t^0[\mathcal{S}^{m-1}(\kappa)]\}.$$

The set of σ in the right-hand term is not empty. If this set coincides with $\boldsymbol{R}^+=\{\sigma;0\leq\sigma<\infty\}$, then $\sigma_j=\infty$.

Let κ^* be the number defined by

$$\kappa^*=\min_{j=1,\cdots,s}\frac{2\rho_j+\sigma_j}{\rho_j-1}$$

where $\kappa^*=2$ if $\rho_j=\infty$ for some j (even though $\sigma_j=\infty$), and $\kappa^*=\infty$ if $\rho_j=1$ for every j. Then the following theorem holds:

Theorem. *Assume the conditions* i)–iii). *Then, if* $1\leq\kappa<\kappa^*$, *the Cauchy problem* (C) *is* $\Gamma^{(\kappa)}$ *well-posed, that is, for any* $\phi_j\in\Gamma^{(\kappa)}$, $j=0,1,\cdots,m-1$, *and for any* $f\in\Gamma^{r(\kappa)}$, *there exists a unique solution* $u(x,t)\in\Gamma^{m+r(\kappa)}$ *of the Cauchy problem* (C); $r=0,1,\cdots,\infty$.

4. Remarks. 1) If $1\leq\kappa<2$, the consequence of the theorem remains true without the assumption iii), whatever the lower order terms of P may be, cf. [2].

2) If $tL_j(\mu_j-\lambda_j)^{-1}\in\mathcal{B}_t^0[\mathcal{S}^{m-2}(\kappa)]$ for every j, then $\kappa^*=\infty$. In this case, even if $\kappa=\kappa^*=\infty$, the Cauchy problem (C) is $\Gamma^{(\infty)}$ well-posed. Here $\Gamma^{(\infty)}=\mathcal{D}_{L^2}^\infty$, $\gamma^{(\infty)}=\mathcal{B}$ and $\mathcal{B}_t^k[\mathcal{S}^r(\infty)]$ is a usual symbol class $\mathcal{B}_t^k[\mathcal{S}^r]$ of pseudo-differential operators. Cf. [5].

3) Assuming the conditions i)–ii), consider the case: *For every* j $(1\leq j\leq s)$, *there exist* $\theta_j>0$ *and* $\delta_j(x,t,\xi)\in\mathcal{B}_t^1[\mathcal{S}^1(\kappa)]$ *such that*

$$\mu_j-\lambda_j=t^{\theta_j}\delta_j,\quad \inf|\delta_j(x,t,\xi)|>0,\quad (x,t)\in\Omega,\quad |\xi|=1.$$

In this case, the condition iii) is automatically satisfied and, whatever the lower order terms of P may be, $\rho_j \leq \theta'_j = \max\{1, \theta_j\}$. Hence, if $1 \leq \kappa < \kappa_1^* = \min_{j=1,\cdots,s} \dfrac{2\theta'_j}{\theta'_j - 1}$, the Cauchy problem (C) is $\Gamma^{(\kappa)}$ well-posed, whatever the lower order terms of P may be.

We will give the proof of the theorem in our forthcoming paper.

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