99. *On a Stability of Essential Spectra of Laplace Operators on Non-Compact Riemannian Manifolds*

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§1. **Introduction.** Let M be an n-dimensional Riemannian manifold, g its Riemannian metric and Δ_g the Laplace operator associated to g. If M is compact, it is well known that Δ_g is essentially selfadjoint in $L_2(M, \mathrm{d}_g x)$, where $\mathrm{d}_g x$ is the volume element associated to g. Also the spectrum $\sigma(\Delta_g)$ of Δ_g consists of only isolated eigenvalues with finite multiplicities. On the other hand, if M is not compact, Δ_g has in general many selfadjoint extensions, and the spectrum may contain continuous part or eigenvalues with infinite multiplicities. In the first case, under a deformation of a Riemannian metric, the eigenvalues move continuously in a certain sense. In this note we concern ourselves with essential spectrum of Δ_g for a non-compact manifold. We show the following

Theorem. *Let (M, g) be a Riemannian manifold. Assume that Δ_g is essentially selfadjoint. Let g_1 be another Riemannian metric which is different from g only on a compact subset K of M. Then,*

(i) *Δ_{g_1} is also essentially selfadjoint in $L_2(M, \mathrm{d}_{g_1} x)$,*

(ii) *the essential spectrum of Δ_g is contained in the spectrum $\sigma(\Delta_{g_1})$ of Δ_{g_1}.*

Here the essential selfadjointness of Δ_g means that the closure $\bar{\Delta}_g$ in $L_2(M, \mathrm{d}_g x)$ of Δ_g acting on $C_0^\infty(M)$ is selfadjoint. In this case, it is easy to show that it coincides with the extension of Δ_g in the sense of distribution, that is, the domain D of $\bar{\Delta}_g$ consists of those $\phi \in L_2(M, \mathrm{d}_g x)$ such that $\Delta_g \phi \in L_2(M, \mathrm{d}_g x)$.

For selfadjoint operators the spectrum can be divided into two parts, the one consisting of all isolated eigenvalues with finite multiplicities and the other, remaining set, called the essential spectrum. The following proposition is known (see [3, p. 518]).

Proposition. *Let λ be in the essential spectrum of a selfadjoint operator A on a Hilbert space H. Then there exists an orthonormal sequence $\{x_n\}_{n\geq 1}$ in H such that*

$$\|(A-\lambda)x_n\| \to 0, \qquad \text{as } n \to \infty.$$

§2. **Proof of the theorem.** For $\phi \in L_2(M, \mathrm{d}_g x)$ we denote its norm by $\|\phi\|$. Let U be an open subset of M such that the closure $\bar{U}$ is compact. We denote by $H_2(U)$ the Sobolev space of degree two. This is defined as follows: Let $\{(U_i, \chi_i)\}$ be a finite number of coordinate systems such that each $\overline{U}_i$ is compact and that the union $\bigcup U_i$ covers $\overline{U}$. Also let $\phi_i \in C_0^\infty(U_i)$ be such that $\phi_i \geqq 0$ and $\bigcup V_i \supset \overline{U}$, where $V_i = \{x \in U_i\,;\, \phi_i(x) > 0\}$. For $\phi \in C_0^\infty(U)$ the $H_2(U)$-norm, $\|\phi\|_{H_2(U)}$, is defined as follows:

$$\|\phi\|^2_{H_2(U)} = \sum \|(\chi_i^{-1})^*(\phi\phi_i)\|^2_{H_2(R^n)}.$$

The space $H_2(U)$ is the completion of $C_0^\infty(U)$ with respect to this norm.

By $(\cdot\,|\,\cdot)_x$ we denote the inner product determined by g in the cotangent space $T_x^*(M)$ at $x \in M$.

Now let us prove the theorem.

The assertion (i) is proved immediately by applying an *a priori* estimate in $H_2(U)$, where $\overline{U}$ is compact and $U \supset K$:

$$\|u\|_{H_2(U)} \leqq C(\|\Delta_g u\| + \|u\|), \qquad u \in H_2(U). \tag{2.1}$$

Let us prove the assertion (ii) in four steps.

I. Let λ be in the essential spectrum of Δ_g. Then by the proposition in § 1, there exists an orthonormal sequence $\{\phi_n\}_{n \geqq 1}$, $\phi_n \in L_2(M, \mathrm{d}_g x)$, such that

$$\|\bar{\Delta}_g \phi_n - \lambda\phi_n\| \to 0, \qquad n \to +\infty. \tag{2.2}$$

II. For $\phi \in C_0^\infty(M)$, let U be an open subset of M such that $U \supset \operatorname{supp}[\phi]$ and $\overline{U}$ is compact. We show that

$$\sup_n \|\phi\phi_n\|_{H_2(U)} < +\infty. \tag{2.3}$$

In fact, we have

$$\bar{\Delta}_g(\phi\phi_n) = \phi\bar{\Delta}_g\phi_n + \phi_n\Delta_g\phi - 2(d\phi_n\,|\,d\phi), \tag{2.4}$$

almost everywhere in M, and $d\phi$ means the exterior derivative of ϕ. Let $\psi \in C_0^\infty(M)$ be such that $\psi = 1$ on a neighborhood V of $\operatorname{supp}[\phi]$ and that $\operatorname{supp}[\psi] \subset U$. Then we have for almost all $x \in M$,

$$|(d\phi\,|\,d\phi_n)_x|^2 \leqq (d\phi\,|\,d\phi)_x(d\phi_n\,|\,d\phi_n)_x = (d\phi\,|\,d\phi)_x(d\phi_n\,|\,d(\psi\phi_n))_x, \tag{2.5}$$

because the first inequality is just the Schwarz inequality, and for the second equality, $(\psi\phi_n)(x) = \phi_n(x)$ for $x \in V$, and all terms are zero for $x \notin \operatorname{supp}[\phi]$. Thus we get from (2.4) and (2.5) the following inequality

$$\begin{aligned} \|\bar{\Delta}_g(\phi\phi_n)\| \leqq {} & \|\bar{\Delta}_g\phi_n\| \sup|\phi| + \|\phi_n\| \sup|\Delta_g\phi| \\ & + 2\|\bar{\Delta}_g\phi_n\|\,\|\phi_n\| \sup(d\phi|d\phi)_x^{1/2}. \end{aligned} \tag{2.6}$$

From this and (2.1) we can see that (2.3) holds.

III. Let $\chi \in C_0^\infty(M)$ be such that $\chi = 1$ on the compact set K, and we assume that ϕ in II is equal to 1 on $\operatorname{supp}[\chi]$. Since the injection from $H_2(U)$ into $L_2(M, \mathrm{d}_g x)$ is compact, we see from (2.3) that there exists a subsequence $\{\phi\phi_{n_k}\}$ of $\{\phi\phi_n\}$ such that

$$\|\phi\phi_{n_k}\| \to 0 \qquad (n_k \to \infty), \tag{2.7}$$

and that for some $\delta_0 > 0$

$$2 \geqq \|(1-\chi)\phi_{n_k}\| > \delta_0. \tag{2.8}$$

We show

(2.9) $$\|(\bar{\Delta}_g-\lambda)(\chi\phi_{n_k})\|\to 0 \qquad (n_k\to\infty).$$

In fact, we have as in II,

(2.10) $$\|(\bar{\Delta}_g-\lambda)(\chi\phi_{n_k})\|\leqq\|(\bar{\Delta}_g-\lambda)\phi_{n_k}\|\sup|\chi|+\|\phi\phi_{n_k}\|\sup|(\Delta_g-\lambda)(\chi)| \\ +2\,\|\bar{\Delta}_g(\phi\phi_{n_k})\|\,\|\phi\phi_{n_k}\|\sup(d\psi|d\psi)^{1/2}.$$

From this together with (2.2), (2.4), (2.6) and (2.7) we see that (2.9) holds.

IV. Finally, since $\chi=1$ on the compact set K,

(2.11) $$\|(\bar{\Delta}_g-\lambda)[(1-\chi)\phi_{n_k}]\|=\|(\bar{\Delta}_{g_1}-\lambda)[(1-\chi)\phi_{n_k}]\|\to 0.$$

This shows that $(1-\chi)\phi_{n_k}$ is in the domain of $\bar{\Delta}_{g_1}$ and with (2.8) λ is not contained in the resolvent set of $\bar{\Delta}_{g_1}$.

Example. For $M=\boldsymbol{R}^n$ with g standard, it is known that Δ_g has only continuous spectrum and $\sigma(\Delta_g)=\boldsymbol{R}_+$. So, under any change of g on a bounded set, the spectrum remains to be $\boldsymbol{R}_+$, because of the positive definiteness of Laplace operators.

§3. Essential selfadjointness of Δ_g. In this section, as an example which satisfies the assumption in our theorem, we show an outline of a proof of the essential selfadjointness of Δ_g, when M is complete.

First we introduce notations.

(1) $L_2=L_2(M,\mathrm{d}_g x)$, L_2^1: the Hilbert space of 1-forms, its inner product is $(\cdot,\cdot)=\int_M(\cdot|\cdot)_x \mathrm{d}_g x$ and we denote the norm by $\|\cdot\|$.

(2) $A_1=\{\phi\,;\,\phi\in C^\infty(M)\cap L_2,\ d\phi\in L_2^1\}$.

(3) $$\begin{cases} B_1=\{\phi\,;\,\phi \text{ is a smooth 1-form, } \phi\in L_2^1,\ \delta\phi\in L_2\}, \text{ where } \delta \text{ is the} \\ \qquad \text{formal adjoint of } d. \\ B_0=\{\phi\,;\,\phi \text{ is a smooth, compactly supported 1-form}\}. \end{cases}$$

(4) $$\begin{cases} D=\{\phi\,;\,\phi\in L_2\cap C^\infty(M),\ \Delta_g\phi\in L_2\}. \\ D_1=\{\phi\,;\,\phi\in L_2,\ \Delta_g\phi\in L_2,\ d\phi\in L_2^1\}. \\ D_w=\{\phi\,;\,\phi\in L_2,\ \Delta_g\phi\in L_2\}. \end{cases}$$

(5) $$\begin{cases} d_0 \text{ is } d \text{ acting on } C_0^\infty(M). \\ d_1 \text{ is } d \text{ acting on } A_1. \\ \delta_0 \text{ is } \delta \text{ acting on } B_0. \\ \delta_1 \text{ is } \delta \text{ acting on } B_1. \end{cases} \qquad \begin{cases} \Delta_0 \text{ is } \Delta_g \text{ acting on } C_0^\infty(M). \\ \Delta \text{ is } \Delta_g \text{ acting on } D. \\ \Delta_1 \text{ is } \Delta_g \text{ acting on } D_1. \\ \Delta_w \text{ is } \Delta_g \text{ acting on } D_w. \end{cases}$$

The proof is done by showing the following (a)–(e).

(a) $\bar{d}_0=\bar{d}_1$ and $\bar{\delta}_0=\bar{\delta}_1$, where $^-$ means the closure of each operator.

(b) $d_1^*=\bar{\delta}_1$, where d_1^* is the adjoint operator of d_1.

(c) $\Delta_1=\bar{\delta}_1\bar{d}_1$, and so Δ_1 is selfadjoint.

(d) $\Delta_1=\bar{\Delta}$.

(e) $\bar{\Delta}=\Delta_w$.

These show that Δ_w is selfadjoint. Also we have $\Delta_w=\Delta_0^*$, and so $\Delta_w^*=(\Delta_0^*)^*$, that is, $\Delta_w=\bar{\Delta}_0$. This means that Δ_0 is essentially selfadjoint.

To prove (a)–(e), we use the following two lemmas.

Lemma 1. *Let x_0 be a fixed point of M, and $\rho(x)$ the geodesic distance from x_0 to x. Then,*

(i) *$\rho(x)$ is a locally Lipschitz continuous function,*

(ii) *for almost all $x \in M$, $(d\rho | d\rho)_x \leqq \dim M$.*

Lemma 2. *Assume that (M, g) is complete. Then for $\phi \in D_w$*

$$\|d\phi\|^2 \leqq (\|\Delta_w \phi\|^2 + \|\phi\|^2).$$

(This is called "Stampacchia's inequality", see [2, Proposition 3, p. 322].)

Now, (a) is proved by making use of Lemma 1 and the completeness of M. (b) was shown in [1] under a general condition, which is satisfied because of (a). (c) is deduced from (b). (d) is a direct consequence of Lemma 2. (e) is a standard property of elliptic operators.

References

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