# 89. *On the Essential Boundary and Supports of Harmonic Measures for the Heat Equation*

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**1.** For the heat equation on an arbitrary bounded domain $D$ located in the $(n+1)$-dimensional Euclidean space $R^{n+1}(=R^n\times R)$ $(n\geqq 1)$, we can solve the Dirichlet problem in the sense of Perron-Wiener-Brelot. Because of this, there exists the harmonic measure $\omega_p$ on the boundary $\partial D$ for every $p\in D$. The support $\operatorname{supp}(\omega_p)$ of $\omega_p$, however, does not coincide with $\partial D$, so the Dirichlet problem and the minimum principle should be considered on a relevant part of the boundary. From the standpoint of the Dirichlet problem, an intrinsic part of the boundary would be $\overline{\bigcup_{p\in D}\operatorname{supp}(\omega_p)}$. This is also available for the minimum principle of superharmonic functions (see Corollary 9).

On the other hand, for the heat equation two kinds of significant boundary part are known. One is the parabolic boundary $\partial_p D$ and the other is the essential boundary $\operatorname{ess}(\partial D)$ (see Definitions 2 and 3).

Our purpose of this paper is to show that our relevant parts $\overline{\bigcup_{p\in D}\operatorname{supp}(\omega_p)}$, $\operatorname{ess}(\partial D)$ and $\overline{\partial_p D}$ are equal except for negligible sets. This throws light on a geometrical property of $\overline{\bigcup_{p\in D}\operatorname{supp}(\omega_p)}$.

**Theorem 1.** (1) $\operatorname{ess}(\partial D)=\overline{\partial_p D}$.

(2) $\operatorname{ess}(\partial D)\supset\overline{\bigcup_{p\in D}\operatorname{supp}(\omega_p)}$ *and* $Z=\operatorname{ess}(\partial D)\setminus\overline{\bigcup_{p\in D}\operatorname{supp}(\omega_p)}$ *is polar.*[1] *Furthermore* $Z\subset D^*\setminus D$, *where* $D^*$ *is the interior of the closure of* $D$.[2]

**2.** Since a bounded domain $D$ associated with the heat equation is a Bauer harmonic space, we follow C. Constantinescu and A. Cornea [1] for basic notation and terminology in potential theory.

We denote by $(x,t)$ a point $p$ in $R^{n+1}$, where $x=(x_1,\cdots,x_n)$ are the space variables and $t$ the time variable.

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1) This means that there exists a positive measure $\mu$ on $R^{n+1}$ such that the potential $G\mu(p)=\int G(p,q)d\mu(q)$ on $R^{n+1}$ takes the value $+\infty$ on $Z$, where for $p=(x,t)$, $q=(y,s)$

$$G(p,q)=\begin{cases}(4\pi(t-s))^{-n/2}\exp(-\|x-y\|/4(t-s)) & \text{if } t>s\\ 0 & \text{if } t\leqq s,\end{cases}$$

and $\|x\|$ denotes the Euclidean norm of $x\in R^n$.

2) This is an affirmative solution of a question proposed orally by Professor M. Itô.

**Definition 2.** For an open set $E$ in $R^{n+1}$, the parabolic boundary $\partial_p E$ is the set of points of $\partial E$ which can be connected to some point of $E$ by a closed curve having strictly increasing $t$-coordinate.

For $p_0 \in E$, we denote by $\Lambda(p_0, E)$ the set of all points $p \in E \setminus \{p_0\}$ which can be connected to $p_0$ by a polygonal line in $E$ having strictly increasing $t$-coordinate. In particular, if $B = B(p_0, r)$ is an open ball with center $p_0 = (x_0, t_0)$ and radius $r > 0$, then $\Lambda(p_0, B)$ is the open half-ball $\{(x, t); \|x - x_0\|^2 + (t - t_0)^2 < r^2, t < t_0\}$.

**Definition 3** (cf. [2, p. 252]). Let $E$ be an open set and put

$$\operatorname{ess}(\partial E) = \{q \in \partial E; \Lambda(q, B(q, \varepsilon)) \neq B(q, \varepsilon) \cap E \text{ for all } \varepsilon > 0\}.$$

We call it the essential boundary of $E$.

Note that $\operatorname{ess}(\partial E)$ is closed.

For a continuous function $f$ on the boundary of a bounded domain $D$, we denote by $H_f^D$ the solution of the Dirichlet problem with boundary condition $f$, which is represented by

$$H_f^D(p) = \int f d\omega_p,$$

where $\omega_p$ is the harmonic measure on $\partial D$ for $p \in D$.

**Lemma 3.** *Let $p_0 = (x_0, t_0) \in D$ and $\Lambda = \Lambda(p_0, D)$. Then*

(1) $\operatorname{supp}(\omega_{p_0}) \subset \{(x, t) \in \partial D; t \leqq t_0\}$.

(2) $\operatorname{supp}(\omega_{p_0}) \subset \operatorname{ess}(\partial D)$.

(3) $\operatorname{supp}(\omega_{p_0}) = \overline{\bigcup_{p \in \Lambda} \operatorname{supp}(\omega_p)}$.

(4) $\omega_p = \psi_p^{p_0}$ *for any* $p \in \Lambda$.

*Here $\omega_p^{p_0}$ denotes the harmonic measure for $p$ with respect to the domain $\Lambda$.*

**Proof.** (1) The potential $u(p) = G(p, p_0)$ on $R^{n+1}$ (see footnote[1]) is strictly positive on $\Lambda(p_0, R^{n+1})$ and zero on $\complement \Lambda(p_0, R^{n+1})$, which implies (1).

(2) Let $q \in \partial D \setminus \operatorname{ess}(\partial D)$; then there exists a ball $B = B(q, r)$ such that $\Lambda(q, B) = B \cap D$. For a continuous function $f$ on $\partial D$, we put $g = f$ on $\partial D \cap B$ and $= H_f^D$ on $\partial B \cap D$. Then

$$H_g^{D \setminus B} = H_f^D \quad \text{on } D \setminus \bar{B}. \tag{2.1}$$

In particular, if $\operatorname{supp}(f) \subset B \cap \partial D$, then by (1), we have $H_f^D(p) = 0$ for any $p \in \bar{B} \cap D$. This and (2.1) imply $H_f^D \equiv 0$, which shows $q \notin \operatorname{supp}(\omega_{p_0})$ and we obtain (2).

(3) By the Harnack inequality,[3] we see that $\omega_p$ is absolutely continuous with respect to $\omega_{p_0}$ for any $p \in \Lambda$, which implies $\operatorname{supp}(\omega_p) \subset \operatorname{supp}(\omega_{p_0})$. On the other hand, for any harmonic function $h$ on $D$, $h(p_0) = 0$ if $h = 0$ on $\Lambda$. Thus, we obtain (3).

(4) Let $f$ be a continuous function on $\partial D$. Putting $g = f$ on $\partial \Lambda \cap \partial D$ and $= H_f^D$ on $\partial \Lambda \cap D$, we have $H_f^D = H_g^\Lambda$ on $\Lambda$. Since $\operatorname{ess}(\partial D)$

3) See [2, Lemma 2].

$\supset \mathrm{ess}(\partial \varLambda)$,[4] $\partial \varLambda \cap D \subset \partial \varLambda \setminus \mathrm{ess}(\partial \varLambda)$, so $H_g^{\varLambda}$ is uniquely determined by the restriction of $f$ on $\partial \varLambda \cap \partial D$, which is denoted by $\tilde{f}$. Hence, putting $\tilde{f}=0$ on $\partial D \setminus \partial \varLambda$ we have $H_{\tilde{f}}^{D}=H_{\tilde{f}}^{\varLambda}$ on $\varLambda$, which shows (4).

We describe the following four lemmas which are known on harmonic spaces.

**Lemma 4** ([1, p. 45, Cor. 2.3.2]). *Let $u$ be a hyperharmonic function on a bounded domain $D$. If* $\liminf_{\substack{p\to\partial D\\ p\in D}} u(p)\geqq 0$, *then $u\geqq 0$ in $D$.*

**Lemma 5** ([1, p. 32, Prop. 3.1.2]). *Let $U, U'$ be open sets with $U'\subset U$, and $u, u'$ be hyperharmonic functions on $U$, on $U'$, respectively. Put $u^*=\inf(u, u')$ on $U'$ and $=u$ on $U\setminus U'$. If $u^*$ is lower semi-continuous, then it is hyperharmonic on $U$.*

Let $(p_n)_{n\geqq 1}$ be a sequence of points in $D$ converging to a boundary point $q\in\partial D$ as $n\to\infty$. We say that $(p_n)_{n\geqq 1}$ is regular (resp. non-regular) if $\omega_{p_n}$ converges (resp. does not converge) vaguely to $\varepsilon_q$, the Dirac measure at $q$. A strictly positive hyperharmonic function on $D$ is called a barrier of $(p_n)_{n\geqq 1}$ if it converges to 0 along $(p_n)_{n\geqq 1}$.

**Lemma 6** ([1, p. 56, Prop. 2.4.7]). *A sequence $(p_n)_{n\geqq 1}$ in $D$ is regular if a barrier of $(p_n)_{n\geqq 1}$ exists.*

Let $v$ be a potential on $D$. A positive hyperharmonic function $u$ on $D$ is called an Evans function of $v$ if for any $c>0$, the set $\{p\in D;\ u(p)\leqq cv(p)\}$ is relatively compact in $D$.

**Lemma 7** ([1, p. 41, Prop. 2.2.4]). *Let $v$ be a continuous potential on $D$. Then there exists an Evans function $u$ of $v$ which is a continuous potential.*

We put

$$R(\partial D)=\left\{\begin{matrix} q\in\partial D; & \text{there exists a regular sequence which} \\ & \text{converges to } q \end{matrix}\right\}.$$

Note that $R(\partial D)$ is closed.

Let $v_0$ be a strictly positive continuous potential on $D$ and $u_0$ be an Evans function of $v_0$ in Lemma 7. In the sequel, $v_0$ and $u_0$ are fixed.

**Proposition 8.** $R(\partial D)=\overline{\bigcup_{p\in D}\mathrm{supp}(\omega_p)}$.

**Proof.** Let $q\notin\overline{\bigcup_{p\in D}\mathrm{supp}(\omega_p)}$; then there exists a ball $B=B(q,r)$ such that $\partial D\cap B\subset \complement\,\overline{\bigcup_{p\in D}\mathrm{supp}(\omega_p)}$. For any continuous function $f$ on $\partial D$ with $\mathrm{supp}(f)\subset\partial D\cap B$ and $f(q)=1$, $H_f^D\equiv 0$. This implies $q\notin R(\partial D)$, that is, $R(\partial D)\subset\overline{\bigcup_{p\in D}\mathrm{supp}(\omega_p)}$.

Next we shall show the inverse inclusion. Let $q_0\notin R(\partial D)$; then there exists a ball $B=B(q_0,r)$ such that $\partial D\cap B\subset\complement R(\partial D)$. We take an arbitrary non-negative continuous function $f$ on $\partial D$ satisfying $\mathrm{supp}(f)\subset\partial D\cap B$. For any $\varepsilon>0$ and any $q\in\partial D$, we shall show

$$\liminf_{\substack{p\to q\\ p\in D}}(\varepsilon u_0(p)-H_f^D(p))\geqq 0. \tag{2.2}$$

4) See [2, Lemma 1].

If $q \in \partial D \cap B$, then, by Lemma 6 $\lim_{\substack{p \to q \\ p \in D}} u_0(p) = +\infty$, which gives (2.2). Suppose that there exist $q$ in $\partial D \backslash B$ and a sequence $(p_n)_{n \geqq 1}$ in $D$ with $\lim_{n \to \infty} p_n = q$ such that

$$\lim_{n \to \infty} (\varepsilon u_0(p_n) - H_f^D(p_n)) < 0. \tag{2.3}$$

Then, $H_f^D$ being bounded on $D$, we see that $(u_0(p_n))_{n \geqq 1}$ is bounded, which implies that $v_0(p_n)$ tends to zero as $n \to \infty$. By Lemma 6, $(p_n)_{n \geqq 1}$ is regular, so $\lim_{n \to \infty} H_f^D(p_n) = f(q) = 0$. This contradicts (2.3) and hence (2.2) holds also for $q \in \partial D \backslash B$. By Lemma 4 we have $\varepsilon u_0 \geqq H_f^D$ in $D$. Letting $\varepsilon \to 0$, we conclude $H_f^D \equiv 0$, which gives $q_0 \notin \overline{\bigcup_{p \in D} \operatorname{supp}(\omega_p)}$. This completes the proof.

**Corollary 9.** *Let $u$ be a superharmonic function on $D$ and assume that $u$ is bounded below. If there exists a constant $A$ such that*

$$\liminf_{\substack{p \to q \\ p \in D}} u(p) \geqq A \qquad \textit{for any } q \in \overline{\bigcup_{p \in D} \operatorname{supp}(\omega_p)}, \tag{2.4}$$

*then $u \geqq A$ in $D$.*

**Proof.** Put $u_n = u + (1/n)u_0 - A$ $(n \geqq 1)$. By Lemma 4, we have $u_n \geqq 0$ in $D$. Letting $n \to \infty$, we obtain Corollary 9.

**Remark 10.** In the above corollary the assumption that $u$ is bounded below is indispensable. For example, let $D_0$ be a bounded domain, $p_0 \in D_0$ and $D = D_0 \backslash \{p_0\}$. Since $\operatorname{supp}(\omega_p) \not\ni p_0$ for any $p \in D$, $\overline{\bigcup_{p \in D} \operatorname{supp}(\omega_p)} \subset \partial D_0$. Hence the harmonic function $u(p) = -G(p, p_0)$ on $D$ satisfies (2.4) for some constant $A$, however, $u \geqq A$ in $D$ does not hold evidently.

**3.** In this paragraph, we shall prove Theorem 1.

**Proof of (1).** We have easily $\partial_p D \subset \operatorname{ess}(\partial D)$. Let $q_0 = (y_0, s_0) \in \partial D \backslash \overline{\partial_p D}$; then there exists a ball $B = B(q_0, r)$ such that $\partial D \cap B \subset \partial D \backslash \overline{\partial_p D}$. For any $p = (x, t) \in D \cap B$, we have $t \leqq s_0$. In fact, let $q_1 = (y_1, s_1)$ be the nearest point from $p$ in the intersection of $\partial D$ and the closed segment $[p, q_0]$. Then $q_1 \in \partial D \cap B$. If $t > s_0$, then $t > s_1$, which contradicts $q_1 \notin \overline{\partial_p D}$. Let $B' = B(q_0, r/2)$ and $q' = (y', s') \in \partial D \cap B'$. Similarly, we obtain $t \leqq s'$ for any $p = (x, t) \in D \cap B$, $s' = s_0$ and $\Lambda(q_0, B') = B' \cap D$. Thus $\operatorname{ess}(\partial D) \subset \overline{\partial_p D}$, that is, $\operatorname{ess}(\partial D) = \overline{\partial_p D}$.

**Proof of (2).** By Lemma 3 and Proposition 8, we see $\operatorname{ess}(\partial D) \supset \overline{\bigcup_{p \in D} \operatorname{supp}(\omega_p)} = R(\partial D)$. Put $K = \operatorname{ess}(\partial D) \backslash R(\partial D)$ and $Z = D^* \cap K$. Then $D \cup Z = D^* \backslash R(\partial D)$, so $D \cup Z$ is a bounded domain. Put $u^* = u_0$ on $D$ and $= +\infty$ on $Z$. Since $u^*$ is lower semi-continuous, Lemma 5 shows that $u^*$ is hyperharmonic on $D \cup Z$. Besides $u^*$ is finite on the dense set of $D \cup Z$, so $u^*$ is superharmonic on $D \cup Z$. The inclusion $Z \subset \{p \in D \cup Z\,;\ u^*(p) = +\infty\}$ implies that $Z$ is polar.[5)]

Next, we shall show that $K = Z$. Suppose $q_0 \in K \backslash Z$; then there

5) From Theorem 25 in [2], it follows that there exists a positive measure $\mu$ on $R^{n+1}$ such that $\{p \in D \cup Z\,;\ u^*(p) = +\infty\} = \{p \in D \cup Z\,;\ G\mu(p) = +\infty\}$.

exists a ball $B=B(q_0, r)$ such that $B\cap\partial D\subset \complement R(\partial D)$ and $B\setminus\bar{D}$ is a non-empty open set. Put $u^{**}=u_0$ on $B\cap D$ and $=+\infty$ on $B\setminus D$. Similarly as above, $u^{**}$ is hyperharmonic on $B$. Since $q_0\in \mathrm{ess}(\partial D)$, we can choose $p_1=(x_1, t_1)\in B\cap D$ and $p_2=(x_2, t_2)\in B\setminus\bar{D}$ such that $t_1>t_2$. This implies that $u^{**}$ is superharmonic on $\Lambda(p_1, B)$,[6] however, $u^{**}$ is infinite on a neighbourhood of $p_2\in\Lambda(p_1, B)$, which is a contradiction, that is, $K=Z$. This completes the proof.

4. In this last paragraph, we shall give a corollary and some remarks to our theorem.

Theorem 1 and Lemma 3 (3), (4) give following

**Corollary 11.** *For any $p_0\in D$, $\mathrm{supp}\,(\omega_{p_0})\subset \mathrm{ess}\,(\partial\Lambda)$, $Z_{p_0}=\mathrm{ess}\,(\partial\Lambda)\setminus \mathrm{supp}\,(\omega_{p_0})$ is polar and $Z_{p_0}\subset\Lambda^*\setminus\Lambda$, where $\Lambda=\Lambda(p_0, D)$.*

**Remark 12.** Theorem 1 is valid for any bounded open set in $R^{n+1}$.

In fact, noting that a countable union of polar sets is polar ([2, Theorem 26]) and applying our theorem to each component of this open set, we have the above remark.

**Remark 13.** For any bounded open set $E$, we have

(1) $\mathrm{ess}\,(\partial E)\supsetneqq\bigcup_{p\in E}\mathrm{supp}\,(\omega_p)$.

(2) $\mathrm{ess}\,(\partial E)\supsetneqq\partial_p E$.

In particular $\bigcup_{p\in E}\mathrm{supp}\,(\omega_p)$ and $\partial_p E$ are not closed.

In fact, let $t_0=\max\{t\,;(x, t)\in\partial E\}$. Since $\partial E\setminus\mathrm{ess}\,(\partial E)$ is open in $\partial E$, there exists a point $p_0\in\mathrm{ess}\,(\partial E)\cap\{(x, t)\in\partial E\,; t=t_0\}$. By Lemma 3 (1), we see that $p_0\notin\bigcup_{p\in E}\mathrm{supp}\,(\omega_p)$. Easily, we see also $p_0\notin\partial_p E$.

**Remark 14.** By the same manner, our theorem is valid not only for heat operator but also for more general parabolic operator, which induces a Bauer harmonic space and possesses the Doob convergence property and has the Green function (cf. [1, p. 95]).

## References

[1] C. Constantinescu and A. Cornea: Potential Theory on Harmonic Spaces. Springer-Verlag, Berlin, Heidelberg, New York (1972).

[2] N. A. Watson: Green function, potential, and the Dirichlet problem for the heat equation. Proc. London Math. Soc., **33** (3), 251–298 (1976).

6) See [2, p. 285].