86. *Polynomial Hamiltonians associated with Painlevé Equations. II**)

Differential equations satisfied by polynomial Hamiltonians

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(Communicated by Kôsaku YOSIDA, M. J. A., Oct. 13, 1980)

1. Introduction. The present article concerns the polynomial Hamiltonians associated with the six Painlevé equations. The notation of the previous note [1] will be adopted throughout this paper; we will refer to the Painlevé equation as P_J ($J=\mathrm{I}, \cdots, \mathrm{VI}$) and denote by H_J the polynomial Hamiltonian $H_J(t; \lambda, \mu)$ associated with P_J, given in Table (**H**) of [1]. Let Ξ_J be the set of fixed critical points of P_J and let $\tilde{\boldsymbol{B}}_J$ be the universal covering surface of $\boldsymbol{B}_J=\boldsymbol{P}^1(\boldsymbol{C})-\Xi_J$. Any solution $(\lambda(t), \mu(t))$ of the Hamiltonian system with the Hamiltonian $H=H_J$,

$$(1)\qquad \begin{cases} \lambda' = \dfrac{\partial H}{\partial \mu} \\ \mu' = -\dfrac{\partial H}{\partial \lambda}, \end{cases}$$

is meromorphic on $\tilde{\boldsymbol{B}}_J$ and so is the function defined by

$$(2)\qquad H_J(t)=H_J(t; \lambda(t), \mu(t)).$$

The τ-function $\tau=\tau_J(t)$ related to $H_J(t)$ is defined by

$$(3)\qquad H_J(t)=\frac{d}{dt}\log \tau_J(t),$$

and it is holomorphic on $\tilde{\boldsymbol{B}}_J$ ([1]).

2. Equation $P_{\mathrm{III}'}$. Consider firstly the equation

$$P_{\mathrm{III}'}\qquad \lambda''=\frac{1}{\lambda}(\lambda')^2-\frac{1}{t}\lambda'+\frac{\lambda^2}{4t^2}(\gamma\lambda+\alpha)+\frac{\beta}{4t}+\frac{\delta}{4\lambda}.$$

We assume that none of γ and δ is zero. In [2], Painlevé showed that $P_{\mathrm{III}'}$ is the limiting form of the equation P_{V} and is transformed to P_{III} by the change of variables: $t\to t^2$, $\lambda\to t\lambda$. Furthermore, we can derive from H_{V} the polynomial Hamiltonian associated with $P_{\mathrm{III}'}$,

$$H_{\mathrm{III}'}\qquad \frac{1}{t}\left[\lambda^2\mu^2-(\eta_\infty\lambda^2+\theta_0\lambda-\eta_0 t)\mu+\frac{1}{2}\eta_\infty(\theta_0+\theta_\infty)\lambda\right],$$

by a process of coalescence. Here the constants in $H_{\mathrm{III}'}$ are related to $\alpha, \beta, \gamma, \delta$ as follows:

$$\alpha=-4\eta_\infty\theta_\infty,\quad \beta=4\eta_0(\theta_0+1),\quad \gamma=4\eta_\infty^2,\quad \delta=-4\eta_0^2.$$

It follows from the assumption $\gamma\delta\neq 0$ that none of η_Δ ($\Delta=0, \infty$) is zero.

*) Partially supported by "The Sakkokai Foundation".

Proposition 1. *System* (1) *with* $\mathrm{H}=\mathrm{H}_{\mathrm{III}'}$ *governs the isomonodromic deformation of the linear equation*

$$\frac{d^2y}{dx^2}+p_1(x:t)\frac{dy}{dx}+p_2(x:t)y=0,$$

where

$$p_1(x:t)=\frac{\eta_0 t}{x^2}+\frac{1-\theta_0}{x}-\eta_\infty-\frac{1}{x-\lambda},$$

$$p_2(x:t)=\frac{\eta_\infty(\theta_0+\theta_\infty)}{2x}-\frac{t\mathrm{H}}{x^2}+\frac{\lambda\mu}{x(x-\lambda)}.$$

It is easy to see that properties of the equation $\mathrm{P}_{\mathrm{III}}$ are derived from those of the equation $\mathrm{P}_{\mathrm{III}'}$, and so we will mainly investigate the equation $\mathrm{P}_{\mathrm{III}'}$. Let $\tau_{\mathrm{III}'}(t)$ be the τ-function related to the function $\mathrm{H}_{\mathrm{III}'}(t)$.

Proposition 2. $\tau_{\mathrm{III}'}(t)$ *is holomorphic on* $\tilde{\boldsymbol{B}}_{\mathrm{III}'}$.

We can suppose, without loss of generality, $\eta_\Delta=1$ by changing scales of t and λ. It will be verified by computation that $\lambda_1=\mu/(\mu-1)$ satisfies the equation P_{V} with

$$\alpha=\frac{1}{8}(\theta_0-\theta_\infty)^2,\quad \beta=-\frac{1}{8}(\theta_0+\theta_\infty)^2,\quad \gamma=2,\quad \delta=0.$$

This fact leads us to

Proposition 3 (cf. [3], [6]). *The equation* P_{V} *with* $\delta=0$ *is equivalent to the equation* $\mathrm{P}_{\mathrm{III}'}$ *with* $\gamma\delta\neq 0$.

Remark 1. In the case when $\gamma=\delta=0$, by substituting λ^2 for λ and t^2 for t, we have the equation

$$\lambda''=\frac{1}{\lambda}(\lambda')^2-\frac{1}{t}\lambda'+\frac{\alpha\lambda^3}{2t^2}+\frac{\beta}{2\lambda},$$

that is, the equation $\mathrm{P}_{\mathrm{III}'}$ with $\gamma\to 2\alpha$, $\delta\to 2\beta$.

3. Differential equations satisfied by the Hamiltonians. By the use of System (1), it will be verified that the function $\mathrm{H}_J(t)$ satisfies a non linear differential equation of the second order. The explicit form of this equation E_J is given below in Table (E), where we suppose that $\eta_\Delta\neq 0$. First we introduce the integer $N(J)$ and the auxiliary constants ν_k $(k=1,\cdots,N(J))$ for the equation P_J as follows:

P_{II} $\quad N(\mathrm{II})=1;\quad \nu_1=\alpha+\frac{1}{2};$

$\mathrm{P}_{\mathrm{III}}$ $\quad N(\mathrm{III})=2;\quad \nu_1=\frac{1}{2}(\theta_0+\theta_\infty),\quad \nu_2=\frac{1}{2}(\theta_0-\theta_\infty),$

$$\bar{\nu}=\left(\frac{1}{2}+\nu_1+\nu_2\right)\left(\frac{1}{2}-\nu_1+\nu_2\right);$$

$\mathrm{P}_{\mathrm{III}'}$ $\quad N(\mathrm{III}')=2;\quad \nu_1=\frac{1}{2}(\theta_0+\theta_\infty),\quad \nu_2=\frac{1}{2}(\theta_0-\theta_\infty);$

P_{IV} $\quad N(\mathrm{IV})=2;\quad \nu_1=\theta_0,\quad \nu_2=\theta_\infty;$

$\mathrm{P_V}$ $\quad N(\mathrm{V})=3$; $\quad \nu_1=\theta_0,\quad \nu_2=\frac{1}{2}(\theta_0+\theta_1+\theta_\infty),\quad \nu_3=\frac{1}{2}(\theta_0+\theta_1-\theta_\infty)$;

$\mathrm{P_{VI}}$ $\quad N(\mathrm{IV})=4$; $\quad \nu_1=\frac{1}{2}(\theta_0+\theta_1),\quad \nu_2=\frac{1}{2}(\theta_0-\theta_1),$

$\nu_3=\frac{1}{2}(\theta_t-1+\theta_\infty),\quad \nu_4=\frac{1}{2}(\theta_t-1-\theta_\infty),$

$\sigma_j(\nu)=$ *the j-th elementary symmetric polynomial of* ν_1,ν_2,ν_3,ν_4,

$\sigma_j^0(\nu)=$ *that of* ν_1,ν_3,ν_4.

Table (E):

$\mathrm{P_I}$ $\quad h=H_{\mathrm{I}}(t),$

$\mathrm{E_I}$ $\quad (h'')^2+4(h')^3+2(th'-h)=0:$

$\mathrm{P_{II}}$ $\quad h=\mathrm{H_{II}}(t),$

$\mathrm{E_{II}}$ $\quad (h'')^2+4(h')^3+2h'(th'-h)-\left(\frac{1}{2}\nu_1\right)^2=0:$

$\mathrm{P_{III}}$ $\quad h=t\cdot\mathrm{H_{III}}(t)+\left(\frac{1}{2}+\nu_1+\nu_2\right)^2,$

$\mathrm{E_{III}}$ $\quad [(th'')^2+4(th'-h)\{(h')^2-16\eta_0\eta_\infty(th'-h-\bar{\nu})\}]^2$
$\qquad\qquad +16^3\eta_0^2\eta_\infty^2(1+2\nu_2)^2(th'-h)^3=0:$

$\mathrm{P_{III'}}$ $\quad h=t\cdot\mathrm{H_{III'}}(t),$

$\mathrm{E_{III'}}$ $\quad (th'')^2-[(\nu_1+\nu_2)h'-\eta_0\eta_\infty\nu_1]^2+4h'(h'-\eta_0\eta_\infty)(th'-h)=0:$

$\mathrm{P_{IV}}$ $\quad h=\mathrm{H_{IV}}(t),$

$\mathrm{E_{IV}}$ $\quad (h'')^2-4(th'-h)^2+4h'(h'+2\nu_1)(h'+2\nu_2)=0:$

$\mathrm{P_V}$ $\quad h=t\cdot\mathrm{H_V}(t)+\nu_2\nu_3,$

$\mathrm{E_V}$ $\quad (\eta_1 th'')^2-[\eta_1^2(th'-h)-2(h')^2-\eta_1(\nu_1+\nu_2+\nu_3)h']^2$
$\qquad\qquad +4h'(h'+\eta_1\nu_1)(h'+\eta_1\nu_2)(h'+\eta_1\nu_3)=0:$

$\mathrm{P_{VI}}$ $\quad h=t(t-1)\cdot\mathrm{H_{VI}}(t)+\sigma_2^0(\nu)t-\frac{1}{2}\sigma_2(\nu),$

$\mathrm{E_{VI}}$ $\quad h'[t(t-1)h'']^2+[h'\{2h-(2t-1)h'\}+\sigma_4(\nu)]^2=\prod_{k=1}^{4}(h'+\nu_k^2).$

We can represent a solution $(\lambda(t),\mu(t))$ of System (1) with $\mathrm{H}=\mathrm{H}_J$ by the function $h=h(t)$ and its derivatives; in fact we have the following

Table (R):

$\mathrm{P_I}$ $\quad \lambda=-h',\quad \mu=-h'':$

$\mathrm{P_{II}}$ $\quad \lambda=\frac{2h''+\nu_1}{4h'},\quad \mu=-2h':$

$\mathrm{P_{III'}}$ $\quad \lambda=4\eta_0\cdot\frac{h-th'+(1/2+\nu_1+\nu_2)\sqrt{h-th'}}{h'\sqrt{h-th'}-th''}$

$\mu=\frac{1}{4\eta_0}\cdot\frac{h'\sqrt{h-th'}-th''}{2\sqrt{h-th'}}:$

$\mathrm{P_{III'}}$ $\quad \lambda=-\frac{\eta_0[th''+\eta_0\eta_\infty\nu_1-(\nu_1+\nu_2)h']}{2h'(h'-\eta_0\eta_\infty)},\quad \mu=\frac{1}{\eta_0}h':$

P_{IV} $\lambda=\dfrac{h''-2(th'-h)}{2(h'+2\nu_2)}$, $\mu=\dfrac{h''+2(th'-h)}{4(h'+2\nu_1)}$:

P_V $\lambda=\dfrac{\eta_1 th''-\eta_1^2(th'-h)+2(h')^2+\eta_1(\nu_1+\nu_2+\nu_3)h'}{2(h'+\eta_1\nu_2)(h'+\eta_1\nu_3)}$,

$$\mu=\frac{\eta_1 th''+\eta_1^2(th'-h)-2(h')^2-\eta_1(\nu_1+\nu_2+\nu_3)h'}{2\eta_1(h'+\eta_1\nu_1)}:$$

P_{VI} $\lambda=\dfrac{1}{2A}\cdot[(\nu_3+\nu_4)B+(h'-\nu_3\nu_4)C]$,

$$\lambda(\lambda-1)\mu=\frac{1}{2A}\cdot[-(h'-\sigma_2^0(\nu))B+(\sigma_1^0(\nu)h'-\sigma_3^0(\nu))C],$$
$$A=(h'+\nu_3^2)(h'+\nu_4^2),$$
$$B=t(t-1)h''+\sigma_1(\nu)h'-\sigma_3(\nu),$$
$$C=2(th'-h).$$

By means of this table, we obtain from a solution $h(t)$ of the non linear differential equation E_J a pair of functions $(\lambda(t), \mu(t))$, which is a solution of System (1) with the Hamiltonian $H=H_J$. Therefore, according to (3) we arrive at

Theorem 1. *$\tau_J(t)$ satisfies a non linear differential equation of the third order and reciprocally a solution $(\lambda(t), \mu(t))$ of System* (1) *are determined by this function and its derivatives.*

Remark 2. Putting for the equation P_{III}

$$g=h+\lambda\mu-\left(\frac{1}{2}+\nu_1+\nu_2\right)^2,$$

we obtain the following expressions;

$$(tg''-g')^2-4[(\nu_1+\nu_2)g'-4\eta_0\eta_\infty\nu_1 t]^2=g'(g'-8\eta_0\eta_\infty t)(4g-2tg'),$$
$$\lambda=-4\eta_0\cdot\frac{(1/2)tg''-(1/2+\nu_1+\nu_2)g'+8\eta_0\eta_\infty\nu_1 t}{g'-8\eta_0\eta_\infty},\qquad \mu=\frac{1}{4\eta_0}\cdot g'.$$

4. Representation of $\lambda(t)$. Now we state the theorem:

Theorem 2. *For $P_{II},\cdots,P_{VI}$, there exist rational functions, $R_i(t;\lambda,\lambda')$ $(i=1,2)$ of (t,λ,λ') and $a(t), b(t)$ of t such that*

(i) *for any solution $\lambda(t)$ of P_J, the functions*

$$\tau_i(t)=\exp\int^t R_i\left(s;\lambda(s),\frac{d\lambda}{ds}(s)\right)ds \qquad (i=1,2)$$

are holomorphic on $\tilde{\boldsymbol{B}}_J$;

(ii) *$a(t), b(t)$ are holomorphic on $\boldsymbol{B}_J$ and*

$$a(t)\lambda(t)+b(t)=\frac{d}{dt}\log\frac{\tau_2(t)}{\tau_1(t)}. \tag{4}$$

This fact was firstly remarked by P. Painlevé [5] for P_{II} and P_{III} without using the Hamiltonian structure. A solution $\lambda(t)$ of P_J and the corresponding τ-function $\tau(t)$ depend on the constants $\nu=(\nu_k)$ $(k=1,\cdots,N(J))$ and $\eta=(\eta_\Delta)$ $(\Delta=0,\infty,1)$. For simplicity of notation, we represent this dependence by $\tau(\nu;\eta)$, $\lambda(\nu;\eta)$. We can prove Theorem

2 by taking as $\tau_i(t)$ two τ-functions of P_J with different values of parameters and as $R_i(t;\lambda,\lambda')$ polynomial Hamiltonians of the corresponding equation. In fact the expression (4) for $\lambda(t)=\lambda(\nu;\eta)$ is given as follows:

P_J	$a(t)$	$b(t)$	$\tau_1(t)$	$\tau_2(t)$
II	1	0	$\tau(\nu_1)$	$\tau(\nu_1-1)$
III	$2\eta_\infty$	$4\eta_0\eta_\infty t^2$	$\tau(\nu_1,\nu_2;\eta_0,\eta_\infty)$	$\tau(\nu_2,\nu_1;\eta_0,-\eta_\infty)$
III′	$\frac{\eta_\infty}{t}$	$\frac{\nu_2-\nu_1}{t}$	$\tau(\nu_1,-\nu_2-1;\eta)$	$\tau(\nu_1+1,-\nu_2;\eta)$
IV	1	0	$\tau(\nu_1,\nu_2)$	$\tau(\nu_1,\nu_2+1)$
V	$\frac{\nu_2-\nu_3}{t}$	0	$\tau(\nu_1,\nu_2,\nu_3+1;\eta_1)$	$\tau(\nu_1,\nu_2+1,\nu_3;\eta_1)$
VI	$\frac{\nu_3-\nu_4}{t(t-1)}$	0	$\tau(\nu_1,\nu_2,\nu_3+1,\nu_4)$	$\tau(\nu_1,\nu_2,\nu_3,\nu_4+1)$

Remark 3. We obtain the following expressions for P_{III}, $P_{III'}$, P_V and P_{VI}:

$$P_{III}\quad 2\eta_\infty\lambda(\nu_1,\nu_2;\eta)+\frac{2\eta_0}{\lambda(\nu_1,\nu_2;\eta)}=\frac{d}{dt}\log\frac{\tau(\nu_1+1,\nu_2;\eta)}{\tau(\nu_1,\nu_2;\eta)};$$

$$P_{III'}\quad \frac{\eta_0}{\lambda(\nu_1,\nu_2;\eta)}-\frac{\nu_1+\nu_2+1}{t}=\frac{d}{dt}\log\frac{\tau(\nu_1+1,\nu_2+1;\eta)}{\tau(\nu_1,\nu_2;\eta)};$$

$$P_V\quad \frac{\eta_1}{1-\lambda(\nu_1,\nu_2,\nu_3;\eta_1)}-\nu_1+\nu_2+\nu_2+1=\frac{d}{dt}\log\frac{\tau(\nu_1,\nu_2+1,\nu_3+1;\eta_1)}{\tau(\nu_1,\nu_2,\nu_3;\eta_1)};$$

$$P_{VI}\quad \frac{\nu_3+\nu_4+1}{t-\lambda(\nu_1,\nu_2,\nu_3,\nu_4)}+c(t)=\frac{d}{dt}\log\frac{\tau(\nu_1,\nu_2,\nu_3+1,\nu_4+1)}{\tau(\nu_1,\nu_2,\nu_3,\nu_4)},$$

$$c(t)=\frac{\nu_1+\nu_2+\nu_3+\nu_4+1}{t}+\frac{\nu_1-\nu_2+\nu_3+\nu_4+1}{t-1}.$$

Remark 4. In [4], another representation of a solution $\lambda(t)$ by the use of τ-functions is given for each of the equations P_J.

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