20. On Excessive Functions

By L. STOICA
Department of Mathematics, INCREST,*) Bucharest

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It was pointed out by T. Watanabe [4, II] that Dynkin's criterion of excessiveness of a function f, is sometimes inconvenient for applications, because it requires two strong conditions:

1) the function f is finely continuous,

2) the function f is supermedian with respect to a very large family of sets.

As an alternative of Dynkin's criterion, Watanabe proved another criterion, in which he replaced the condition 1) with the stronger one, that f was lower semicontinuous, while condition 2) was weakened by considering a family $\mathcal{U}$ that had to be only a base. Furthermore it was conjectured that in this criterion the lower semicontinuity of f can be replaced by a weaker continuity condition stated in terms of $\mathcal{U}$.

Here we give a positive answer to this conjecture, in the case of an instantaneous state process. A version of this criterion is very useful in the case of a Markov process associated to an elliptic strongly degenerated differential operator [3].

Let E be a locally compact space with a countable open base and $\mathcal{E}$ the σ-algebra of Borel sets of E. Further let $(\Omega, \mathcal{M}, \mathcal{M}_t, X_t, \theta_t, P^x)$ be a standard process with state space $(E, \mathcal{E})$. For notations and definitions in the Markov process theory we refer to [1].

If A is a nearly Borel set, $f \in \mathcal{E}_+$ and $x \in E$ we denote $E^x[f(x_{T_{CA}})]$ by $H^A f(x)$.

Suppose that $\mathcal{U}$ is a family of nearly Borel sets such that for each point $x \in E$ and each neighbourhood V of x there exists $U \in \mathcal{U}$, $x \in \mathring{U}$, $U \subset V$. For any $x \in E$ the family $\mathcal{U}(x) = \{U \in \mathcal{U} / x \in \mathring{U}\}$ becomes a directed set under the order relation "$U_1 \leqslant U_2$ if $U_2 \subset \mathring{U}_1$".

Theorem. *If $s: E \to \bar{R}_+$ is an universally measurable function such that:*

(a) $H^U s \leqslant s$ *for any* $U \in \mathcal{U}$,

(b) $s(x) = \lim_{U \in \mathcal{U}(x)} H^U s(x)$ *for any* $x \in E$,

then s is excessive.

Proof. We consider a metric d on E and for each fixed $n \in N$,

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*) Institutul Național Pentru Creație Științifică Și Techņică.

$n\geqslant 1$, choose a sequence $\{D_i/i\in N\}$ of open sets and another sequence $\{U_i/i\in N\}\subset\mathcal{U}$ such that

$$\bigcup_{i\in N} D_i=E,\quad \bar{D}_i\subset\mathring{U}_i,\quad d(U_i)<1/n,\quad (\forall)i\in N,$$

the set $\{i\in N/U_i\cap K\neq\phi\}$ is finite for any compact set K. We define $R(\omega)=T_{CU_i}(\omega)$ if $X_0(\omega)\in D_i\setminus\bigcup_{j=1}^{i-1}D_j$, then put $R_0=0$, $R_1=R$ and

$$R_{k+1}=R_k+R\circ\theta_{R_k}\qquad\text{for each } k\in N,\ k\geqslant 1.$$

$\{R_k/k\in N\}$ are stopping times (see [4], (II) Lemma 3.2) and $\lim_{k\to\infty}R_k=\xi$. The function $s_n: E\to\bar{R}_+$, defined by $s_n(x)=\inf\{H^{U_i}s(x)/i\in N,\ x\in\mathring{U}_i\}$ is universally measurable ([1], p. 61). Further let $x_0\in E$, $t>0$, $n\in N$, $n\geqslant 1$. We are going to prove the following inequality by induction:

(1) $$s(x_0)\geqslant E^{x_0}[s_n(X_t);\ t\leqslant R_k]+E^{x_0}[s(X_{R_k});\ R_k<t].$$

For $k=0$ it is trivial. Further (a) implies:

(2) $$s(x)\geqslant E^x[s(X_R)].$$

On the other hand we have

$$E^x[s(X_{T_{CU_i}});\ t<T_{CU_i}]=E^x[H^{U_i}s(X_t);\ t<T_{CU_i}]$$
$$\geqslant E^x[s_n(X_t);\ t<T_{CU_i}],$$

and hence $E^x[s(X_R);\ t-r<R]\geqslant E^x[s_n(X_{t-r});\ t-r<R]$.

In this inequality we put $x=X_{R_k}(\omega)$ and $r=R_k(\omega)$ and integrate over $\{\omega/R_k(\omega)<t\}$ with respect to $dP^{x_0}(\omega)$:

(3) $$\int\chi_{\{\omega/R_k(\omega)<t\}}\int s(X_R(\omega'))\cdot\chi_{\{\omega'/t-R_k(\omega)<R(\omega')\}}dP^{X_{R_k}(\omega)}(\omega')dP^{x_0}(\omega)$$
$$\geqslant\int\chi_{\{\omega/R_k(\omega)<t\}}\int s_n(X_{t-R_k(\omega)}(\omega'))\chi_{\{\omega'/t-R_k(\omega)<R(\omega')\}}dP^{X_{R_k}(\omega)}(\omega')dP^{x_0}(\omega).$$

Using the strong Markov property,[1] we can rewrite the last term as

$$E^{x_0}[s_n(X_t);\ t-R_k<R\circ\theta_{R_k};\ R_k<t].$$

Now in (2) we put $X_{R_k}(\omega)$ instead of x and integrate both sides of (2) over $\{\omega/R_k(\omega)<t\}$:

$$E^{x_0}[s(X_{R_k})/R_k<t]\geqslant\int\chi_{\{\omega/R_k(\omega)<t\}}\int s(X_R(\omega'))dP^{X_{R_k}(\omega)}(\omega')dP^{x_0}(\omega),$$

further, using (3) we get

$$\geqslant E^{x_0}[s_n(X_t);\ R_k<t<R_{k+1}]$$
$$+\int\chi_{\{\omega/R_k(\omega)<t\}}\int s(X_R(\omega'))\chi_{\{\omega'/R(\omega')\leqslant t-R_k(\omega)\}}dP^{X_{R_k}(\omega)}(\omega')dP^{x_0}(\omega).$$

Again the strong Markov property[1] shows that the last term equals

$$E^{x_0}[s(X_{R_{k+1}});\ R_{k+1}\leqslant t]$$

Thus we have

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[1] We have used the strong Markov property in the following form: If τ is a stopping time and $G(\omega,\omega')$ an $\mathcal{M}_\tau\otimes\mathcal{F}$ measurable non-negative functions, then $E^x[G(\cdot,\theta_\tau(\cdot));\mathcal{M}_\tau](\omega)=E^{X_\tau(\omega)}[G(\omega,\cdot)]$.

$$(4)\qquad \begin{aligned} &E^{x_0}[s(X_{R_k});R_k<t]\\ &\quad\geqslant E^{x_0}[s_n(X_t);R_k<t<R_{k+1}]+E^{x_0}[s(X_{R_{k+1}});R_{k+1}\leqslant t]\end{aligned}$$

Now let us suppose that (1) is valid; from (1) and (4) wet get

$$\begin{aligned} s(x_0)\geqslant E^{x_0}[s_n(X_t);t<R_{k+1}]+E^{x_0}[s(X_t);t=\boldsymbol{R}_{k+1}]\\ +E^{x_0}[s(X_{R_{k+1}});R_{k+1}<t],\end{aligned}$$

which leads to formula (1) with for $k+1$ instead of k.

Letting $k\to\infty$ we have

$$s(x_0)\geqslant E^{x_0}[s_n(X_t)].$$

But since condition (b) implies $s=\lim_{n\to\infty} s_n$, we obtain

$$s(x_0)\geqslant \liminf_{n\to\infty} E^{x_0}[s_n(X_t)]\geqslant E^{x_0}[s(X_t)].$$

If $U\in\mathcal{U}(x_0)$, then

$$\begin{aligned} s(x_0)\geqslant \limsup_{t\to 0} E^{x_0}[s(X_t)]\geqslant \liminf_{t\to 0} E^{x_0}[s(X_t)]\\ \geqslant \lim_{t\to 0} E^{x_0}[H^U s(X_t);t<T_{CU}]=H^U s(x_0),\end{aligned}$$

and hence $s(x_0)=\lim_{t\to 0} E^{x_0}[s(X_t)]$.

References

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