12. *Limit Circle Type Criteria for Nonlinear Differential Equations*

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In 1910 Weyl [15] published his now classic paper on the classification of solutions of second order linear differential equations into the limit circle and limit point types. He said that the equation

$$(a(t)x')' + q(t)x = 0 \tag{1}$$

is of the limit circle type if all its solutions are square integrable, i.e., if

$$\int^{\infty} x^2(u)du < \infty\ ; \tag{2}$$

otherwise the equation is said to be of the limit point type. In the ensuing years there has been a great deal of interest in the limit point-limit circle problem due to its importance in relation to the solution of certain boundary value problems (see Titchmarsh [13], [14]). An excellent discussion and survey of known results on this problem can be found in the treatise of Dunford and Schwartz [5]. Some recent papers on this problem include those of Burton and Patula [3], Everitt [6], Hinton [9], Knowles [10], Wong [17], and Wong and Zettl [18]; additional references can be found in [1]–[18]. Only in recent years has any progress been made on this problem for nonlinear equations (see [1], [2], [4], [8], [11], [12] and [16]). The purpose of this note is to announce some new integrability results obtained by the author which, when reduced to the case of equation (1), yield the square integrability of all solutions, i.e., the limit circle case as defined by Weyl [15]. Complete details and further discussion of the results presented here will appear in [7].

We consider the second order nonlinear differential equation

$$(a(t)x')' + q(t)f(x) = r(t) \tag{3}$$

where $a, q, r : [t_0, \infty) \to R$ and $f : R \to R$ are continuous, $a(t) > 0$, $q(t) > 0$, and $xf(x) \geq 0$ for all x. We will say that equation (3) is of nonlinear limit circle type if every solution $x(t)$ of (3) satisfies

$$\int_{t_0}^{\infty} x(u)f(x(u))du < \infty, \tag{4}$$

and we will say that equation (3) is of nonlinear limit point type otherwise. This of course reduces to (2) when $f(x) = x$. Discussions of other possible definitions of limit point and limit circle for nonlinear equations can be found in [1] and [7].

Just as other authors have used various transformations of the independent variable t to obtain limit circle and limit point results for equation (1), we too introduce such a transformation to study equation (3). Moreover, the transformation used here does not appear to reduce to one of those used in the study of equation (1). In order to illustrate the technique without becoming unnecessarily involved in details, we consider here a special case of equation (3), namely, the well known Emden-Fowler equation

$$x''+q(t)x^{2n-1}=0 \tag{5}$$

where n is a positive integer, $q'\in AC_{\mathrm{loc}}[t_0,\infty)$, and $q''\in L^2_{\mathrm{loc}}[t_0,\infty)$. Note that in the case of equation (5), condition (4) is equivalent to having all solutions belong to $L^{2n}(t_0,\infty)$.

To simplify the notation we let $\alpha=1/2(n+1)$ and $\beta=(2n+1)/2(n+1)$ so $\beta-\alpha=n/(n+1)$. If we let

$$s=\int_{t_0}^{t} q^{\alpha}(u)du \quad \text{and} \quad y(s)=x(t), \tag{6}$$

then equation (5) becomes

$$\ddot{y}+\alpha p(t)\dot{y}+P(t)y^{2n-1}=0 \tag{7}$$

where "$\cdot$"$=d/ds$, $p(t)=q'(t)/q^{\alpha+1}(t)$, and $P(t)=q^{\beta-\alpha}(t)$. We will write equation (7) as the system

$$\begin{aligned}\dot{y}&=z-\alpha p(t)y\\ \dot{z}&=-P(t)y^{2n-1}+\alpha\dot{p}(t)y.\end{aligned} \tag{8}$$

Theorem 1. *Assume that*

$$\int_{t_0}^{\infty}|[q'(u)/q^{\alpha+1}(u)]'/q^{(\beta-\alpha)/2}(u)|\,du<\infty \tag{9}$$

and

$$\int_{t_0}^{\infty}|[q'(u)/q^{\alpha+1}(u)]'q^{(\beta-\alpha)/2}(u)|\,du<\infty. \tag{10}$$

If

$$\int_{t_0}^{\infty}[1/q^{\beta-\alpha}(u)]du<\infty, \tag{11}$$

then equation (5) *is of nonlinear limit circle type, i.e., all solutions of* (5) *belong to* $L^{2n}(t_0,\infty)$.

Proof. Define $V(y,z,s)=z^2/2+P(t)y^{2n}/2n$. Then $\dot{V}(s)=\alpha\dot{p}(t)yz+(\dot{P}(t)/2n-\alpha p(t)P(t))y^{2n}=\alpha\dot{p}(t)yz$. Now

$$\begin{aligned}|yz|&=|P^{1/2}(t)yz|/P^{1/2}(t)\\ &\le(P(t)y^2/2+z^2/2)/P^{1/2}(t)\\ &\le[P(t)(y^{2n}/2n+K_1)+z^2/2]/P^{1/2}(t)\\ &\le V/P^{1/2}(t)+K_1P^{1/2}(t)\end{aligned}$$

for some constant $K_1\ge 0$. Hence

$$\dot{V}(s)\le\alpha|\dot{p}(t)|\,V(s)/P^{1/2}(t)+\alpha K_1|\dot{p}(t)|\,P^{1/2}(t). \tag{12}$$

Since $\dot{p}(t)=p'(t)/q^{\alpha}(t)$, we have

$$\int_{s_0}^{s}|\dot{p}(\tau(\xi))|\,P^{1/2}(\tau(\xi))d\xi=\int_{t_0}^{t}|p'(u)|\,P^{1/2}(u)du$$

which converges by condition (10). Hence if we integrate (12) from s_0 to s and apply Gronwall's inequality, condition (9) then insures that $V(s)$ is bounded. Thus we have

$$P(t)x^{2n}/2n = P(t)y^{2n}/2n \leq K_2$$

for some constant $K_2 \geq 0$. It follows from condition (11) that all solutions of (5) belong to $L^{2n}(t_0, \infty)$.

If $n=1$ so that equation (5) is linear, we see that the constant K_1 in the proof of the theorem can be chosen to be zero, and so in this case condition (10) is not needed. Note also that when $n=1$, condition (9) is exactly the well known condition

$$\int_{t_0}^{\infty} |q''(u)/q^{3/2}(u) - (5/4)[q'(u)]^2/q^{5/2}(u)|\, du < \infty$$

of Dunford and Schwartz [5, p. 1414].

The following theorem gives sufficient conditions for equation (5) to be of nonlinear limit point type, i.e., at least one solution of (5) does not belong to $L^{2n}(t_0, \infty)$. The details of the proof are much too lengthly to present here but they can be found in [7].

Theorem 2. *Suppose that conditions* (9) *and* (10) *hold, there is a positive constant A such that*

$$|q'(t)/q^{3/2}(t)| \leq A \tag{13}$$

and

$$\int_{t_0}^{\infty} \{[q'(u)]^2/q^3(u)\} du < \infty. \tag{14}$$

If

$$\int_{t_0}^{\infty} [1/q^{\beta-\alpha}(u)] du = \infty,$$

then equation (5) *is of nonlinear limit point type.*

By combining Theorems 1 and 2 we can obtain the following necessary and sufficient condition for equation (5) to be of nonlinear limit circle type.

Corollary 3. *If conditions* (9), (10), (13) *and* (14) *hold, then equation* (5) *is of nonlinear limit circle type if and only if*

$$\int_{t_0}^{\infty} [1/q^{\beta-\alpha}(u)] du < \infty. \tag{15}$$

Condition (15) is sharp since if $q(t)=t^{\sigma}$ then (15) holds if and only if $\sigma n/(n+1) > 1$, i.e., $\sigma > 1 + 1/n$. This is in complete agreement with Atkinson's results [1] and with what is known from asymptotic integrations of equation (5) when $q(t)=t^{\sigma}$. Also, it is possible to improve the integral conditions (9) and (10) slightly for the cases $n \neq 1$ (see [7]). Finally, we note that the relationship between the nonlinear limit circle property and the boundedness, oscillation, and convergence to zero of solutions of (3) is also discussed in [7].

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