71. *J-Compatible Orthodox Semigroups*

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A semigroup S is said to be *J-compatible* if Green's J-relation is a congruence on S. In this paper, we shall study the structure of J-compatible orthodox semigroups.

1. Basic properties. Let S be a regular semigroup. Throughout this paper, the J-relation and the D-relation on S will be denoted by $\mathcal{J}_S$ and $\mathcal{D}_S$ respectively. Further, the congruence generated by $\mathcal{J}_S[\mathcal{D}_S]$ will be denoted by $\mathcal{J}_S^*[\mathcal{D}_S^*]$. Let η_S be the least semilattice congruence on S. Then, it has been shown by Hall [2] that $\mathcal{D}_S^*=\eta_S$. Further, it is easily seen that $\mathcal{D}_S\subset\mathcal{J}_S\subset\eta_S$. In fact, if $(a, b)\in\mathcal{D}_S$ then there exists $c\in S$ such that $a\mathcal{L}_S c\mathcal{R}_S b$, where $\mathcal{L}_S$ and $\mathcal{R}_S$ denote the L-relation and the R-relation on S respectively. Hence, $Sa=Sc$ and $cS=bS$. Accordingly, $SaS=ScS=SbS$. Therefore, $(a, b)\in\mathcal{J}_S$. Since η_S is the least semilattice congruence on S, S is a semilattice Γ of the η_S-classes $\{S_\gamma:\gamma\in\Gamma\}$ (in this case, $\Gamma\cong S/\eta_S$) and each S_γ is semilattice-indecomposable. If $a\mathcal{J}_S b$, then $SaS=SbS$. Hence, there exist $x, y\in S^1$ and $u, v\in S^1$ such that $uav=b$ and $xby=a$. Let $a\in S_\alpha$ and $b\in S_\beta$. Since $uav=b$, it follows that $\beta\leq\alpha$. On the other hand, $\alpha\leq\beta$ follows from $xby=a$. Hence, $\alpha=\beta$. Consequently, $a\eta_S b$. Thus, $\mathcal{J}_S\subset\eta_S$.

Since $\mathcal{D}_S^*=\eta_S$, we have $\mathcal{J}_S^*=\eta_S$. Hence, $\mathcal{D}_S^*=\mathcal{J}_S^*=\eta_S$. In particular, if S is J-compatible then $\mathcal{D}_S^*=\mathcal{J}_S=\eta_S$.

Theorem 1. *For a regular* [*orthodox*] *semigroup S, the following conditions are equivalent*:

(1) *S is J-compatible.*

(2) $\mathcal{J}_S=\eta_S$.

(3) *S is a semilattice of simple regular* [*orthodox*] *semigroups.*

(4) *$J(a)\cap J(b)=J(ab)$ for $a, b\in S$ (where $J(x)=SxS$); hence, the principal ideals of S form a semilattice under intersection.*

Proof. (1)$\Rightarrow$(2): This was already proved above. The part "(2)$\Rightarrow$(3)" follows from Petrich [6, p. 43]. Further, both "(3)$\Rightarrow$(4)" and "(4)$\Rightarrow$(1)" follow from Clifford and Preston [1, p. 123].

If $\mathcal{D}_S^*=S\times S$ for a semigroup S, then S is said to be *D^*-simple.*

Theorem 2. *If a regular semigroup S is simple, then S is D^*-simple.*

Proof. As was shown above, $\mathcal{D}_S^*=\eta_S$. Since S is simple, η_S is the universal relation. Hence, $\mathcal{D}_S^*$ is also the universal relation. That is,

S is D^*-simple.

The following example shows that a D^*-simple regular semigroup is not necessarily simple:

Example. Let $E=\{0, e, f\}$ be a semilattice such that $0e=e0=0f=f0=ef=fe=00=0$, $e^2=e$ and $f^2=f$. Then, E is clearly P-uniform (see [3]). Let $\varepsilon_0, \varepsilon_e, \varepsilon_f$ be the identity mappings on the sets $\{0\}$, $\{e, 0\}$ and $\{f, 0\}$ respectively, and α the mapping of $\{e, 0\}$ onto $\{f, 0\}$ defined by $0\alpha=0$ and $e\alpha=f$. Let I_E be the symmetric inverse semigroup on the set $\{0, e, f\}$, and T_E the inverse subsemigroup of I_E consisting of all τ such that the domain and codomain of τ are principal ideals of E and τ is an isomorphism (see [5]). Then $T_E=\{\varepsilon_0, \varepsilon_e, \varepsilon_f, \alpha, \alpha^{-1}\}$, and it is easily seen from [3] that T_E is a P-transitive inverse subsemigroup of T_E itself. Hence by [3], T_E is D^*-simple. However, T_E is not simple since $\{\varepsilon_0\}$ is a proper ideal of S.

By Theorem 1, the problem of describing all J-compatible orthodox semigroups can be divided into the following two problems:

1) Describing all simple orthodox semigroups.

2) Semilattice-compositions of simple orthodox semigroups.

The second problem is a special case of the semilattice-composition problem for general semigroups (for example, see Petrich [6]), and we do not deal with this problem here any more. On the other hand, it has been shown by [7] and [8] that a simple orthodox semigroup can be obtained as a quasi-direct product of a band and a simple inverse semigroup. Further, it is easily seen from [3] that a simple inverse semigroup can be obtained as a regular (Schreier) extension of a Clifford semigroup (that is, a semilattice of groups) by a fundamental simple inverse semigroup (see also [4]). Therefore, the first problem is reduced to the problem of describing all possible fundamental simple inverse semigroups.

2. *I*-transitive inverse subsemigroups of T_E. Let E be a semilattice, and let I_E be the symmetric inverse semigroup on E. Let T_E be the subset of I_E consisting of all τ in I_E such that the domain $D(\tau)$ and codomain $R(\tau)$ of τ are principal ideals of E and τ is an isomorphism. Then, T_E is an inverse semigroup. For each $e \in E$ the identity automorphism of eE will be denoted by ε_e. The following result has been shown by Munn [5]: *Every full inverse subsemigroup S of T_E is fundamental. The semilattice of idempotents of S is $E^*=\{\varepsilon_e : e \in E\}$ and the mapping $\theta: E \to E^*$ defined by $e\theta=\varepsilon_e$ is an isomorphism.*

Now, let us consider the following conditions for semilattices E:

(2.1) For any proper ideal F of E, there exist elements e, f of E such that $e \in E \backslash F$, $f \in F$ and $eE \cong fE$ (eE is isomorphic to fE).

(2.2) For all $e, f \in E$ there exists $g \in E$ such that $g \leq f$ and $eE \cong gE$.

Lemm 3. *If an inverse semigroup S is simple, then*

(2.3) *for any proper ideal F of E_S (the semilattice of idempotents of S), there exist elements $e, f \in E_S$ and $c \in S$ such that $e \in E_S \backslash F$, $f \in F$, $cc^{-1}=e$, $c^{-1}c=f$ and $\theta_c: eE_S \rightarrow fE_S$ defined by $u\theta_c=c^{-1}uc$ is an isomorphism.*

Accordingly, in this case E_S satisfies the condition (2.1).

Proof. Let S be a simple inverse semigroup. Assume that for some proper ideal F of E_S there is no pair of e, f, c satisfying the condition in the statement. First, we prove that $a^{-1}Ea \subset F$ for any $a \in S$. If not, then there exists $a \in S$ such that $a^{-1}Fa \not\subset F$. Then, there exists $e_1 \in F$ such that $a^{-1}e_1a \in F_S \backslash F$. Put $a^{-1}e_1a=f$. Then, $a^{-1}e_1E_Sa = a^{-1}e_1aa^{-1}E_Sa = fa^{-1}E_Sa \subset fE_S$. Conversely, $fE_S = a^{-1}afE_Sa^{-1}a = a^{-1}aa^{-1}e_1aE_Sa^{-1}a \subset a^{-1}(aa^{-1}e_1E_Sa) \subset a^{-1}e_1E_Sa$. Hence, $a^{-1}e_1E_Sa = fE_S$. Putting $af=c$ and $aa^{-1}e_1=e$, it follows that

$$c^{-1}c=fa^{-1}af=f \in E_S \backslash F, \qquad cc^{-1}=afa^{-1}=aa^{-1}e_1aa^{-1}=e \in F$$

and

$$c^{-1}eE_Sc=fa^{-1}aa^{-1}e_1E_Saf=fa^{-1}e_1E_Saf=fE_S.$$

Now, the mapping $\theta_c: eE_S \rightarrow fE_S$ defined by $(eu)\theta_c=c^{-1}euc$, $u \in E_S$, is an isomorphism. This contradicts to the first assumption. Therefore, $a^{-1}Fa \subset F$ for all $a \in S$. Now, put $C=\{a \in S: aa^{-1}, a^{-1}a \in F\}$. Then, $C \neq S$ and C is an ideal of S. This is proved as follows: For $a \in C$ and for $x \in S$, $ax(ax)^{-1}=axx^{-1}a^{-1}=aa^{-1}axx^{-1}a^{-1} \in aFa^{-1} \subset F$ and $(ax)^{-1}ax = x^{-1}a^{-1}ax \in x^{-1}Fx \subset F$. Hence, C is an ideal. Accordingly, S is not simple.

Lemma 4. *For a fundamental inverse semigroup S, the following conditions are equivalent:*

(1) *S is simple.*

(2) *S satisfies the following:*

(2.4) *For any $e, f \in E_S$ there exists $a \in S$ such that $aa^{-1}=e$, $a^{-1}a = g \leq f$ and $a^{-1}eE_Sa=gE_S$.*

(3) *S satisfies the condition* (2.3).

Proof. (1)$\Rightarrow$(2): This is obvious from the proof of Theorem 3.2, (i), of Munn [5]. (2)$\Rightarrow$(3): This is obvious. (3)$\Rightarrow$(1): Assume that S is not simple. Then, there exists a proper ideal A of S. If $E_A=E_S$, then $cc^{-1} \in E_A$ for any $c \in S$. Hence, $cc^{-1}c \in A$. That is, $c \in A$. Therefore, we have $S=A$. Hence, $E_A \neq E_S$. Since E_A is a proper ideal of E_S, there exists a pair of $e, f, a \in S$ such that $e \in E_S \backslash E_A$, $f \in E_A$, $eE_S \cong fE_S$, $aa^{-1}=e$, $a^{-1}a=f$ and $a^{-1}eE_Sa=fE_S$. Hence, $a=af \in A$. Consequently, $e=aa^{-1} \in A$. That is, $e \in E_A$. This contradicts to $e \in E_S \backslash E_A$. Accordingly, S is simple.

Remark. Munn [5] has given a description of fundamental simple inverse semigroups by using the property (2) of Lemma 4. In this paper, we shall give another description by using the property (3) of Lemma 4.

Let E be a semilattice. According to [5], an inverse subsemigroup S of T_E is said to be *subtransitive* if S satisfies the following:

(2.5) For any $e, f \in E$ there exists $\gamma \in S$ such that $eE = D(\gamma)$ and $R(\gamma) \subset fE$.

Now, we introduce the concept of *an I-transitive inverse subsemigroup of* T_E. An inverse subsemigroup S of T_E is said to be I-transitive if S satisfies the following:

(2.6) For any proper ideal F of E, there exist elements e, f of E and $\gamma \in S$ such that $e \in E \backslash F$, $f \in F$, $eE = D(\gamma)$ and $fE = R(\gamma)$.

Lemma 5. *If a semilattice E satisfies the condition* (2.1),

(1) *every I-transitive full inverse subsemigroup S of T_E is a fundamental simple inverse semigroup,*

(2) *hence, the conditions* (2.1) *and* (2.2) *are equivalent.*

Proof. (1) It is obvious that S is a fundamental inverse semigroup. The semilattice of idempotents of S is $E^* = \{\varepsilon_e : e \in E\}$ and $\theta : E \to E^*$ defined by $e\theta = \varepsilon_e$ is an isomorphism. Let $F^* = \{\varepsilon_e : e \in F\}$ be a proper ideal of E^*. Then, F is of course also a proper ideal of E. Since S is I-transitive, there exist $e \in E \backslash F$, $f \in F$ and $\gamma \in S$ such that $D(\gamma) = eE$, $R(\gamma) = fE$ and $(eE)\gamma = fE$. Hence, $\gamma\gamma^{-1} = \varepsilon_e$, $\gamma^{-1}\gamma = \varepsilon_f$, and $\theta_\gamma : \varepsilon_e E^* \to \varepsilon_f E^*$ defined by $\upsilon\theta_\gamma = \gamma^{-1}\upsilon\gamma$, $\upsilon \in \varepsilon_e E^*$, is an isomorphism. Hence, it follows from Lemma 4 that S is simple.

(2) It is obvious that a semilattice satisfying (2.2) satisfies (2.1). Suppose that a semilattice E satisfies (2.1). Then, by (1) T_E is a fundamental simple inverse semigroup. Hence, by Lemma 4 the semilattice E^* of idempotents of T_E satisfies (2.2). Since E^* is isomorphic to E, E also satisfies (2.2).

Munn [5] calls a semilattice E satisfying (2.2) (hence, (2.1)) *subuniform.*

Finally, we have the following result:

Theorem 6. (1) *Let S be a fundamental simple inverse semigroup. Then, E_S is subuniform and S satisfies* (2.3). *Further, S is isomorphic to an I-transitive full inverse semigroup of T_{E_S}.*

(2) *Conversely, let E be a subuniform semilattice. Then, every I-transitive full inverse subsemigroup of T_E is a fundamental simple inverse semigroup.*

Proof. (1) The first part was already proved in Lemma 3. By Munn [5], the mapping $\theta : S \to T_{E_S}$ defined by $a\theta = \theta_a$, where θ_a is the element of T_{E_S} such that the domain is $aa^{-1}E_S$, the codomain is $a^{-1}aE_S$ and $x\theta_a = a^{-1}xa$ for $x \in aa^{-1}E_S$, is a monomorphism. Since $\theta_e = \varepsilon_e$ for

$e \in E_S$, $S\theta$ is a full inverse subsemigroup of T_{E_S}. Let $E_S^* = \{\varepsilon_e(=\theta_e) : e \in E_S\}$. Then, E_S^* is the semilattice of idempotents of T_{E_S}, and the restriction $\theta | E_S$ of θ to E_S is an isomorphism of E_S onto E_S^*. Let F be a proper ideal of E_S. Since S satisfies (2.3), there exist $e, f \in E_S$ and $a \in S$ such that $e \in E_S \backslash F$, $aa^{-1} = e$, $a^{-1}a = f$ and $a^{-1}eE_S a = fE_S$. Hence, $D(\theta_a) = eE_S$, $R(\theta_a) = fE_S$ and $(eE_S)\theta_a = fE_S$. Therefore, $S\theta$ is I-transitive.

(2) Obvious from Lemma 5.

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