44. A Counter Example of Ampleness of Positive Line Bundles

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0. Let X be a weakly 1-complete manifold and B be a positive line bundle over X. If X is compact or Stein it is well known that B is ample, i.e., there exist an integer ν and sections $s_0, \cdots, s_N \in \Gamma(X, B^\nu)$ such that $(s_0 : \cdots : s_N)$ gives a holomorphic embedding of X into $\boldsymbol{P}^N$. The purpose of this note is to show that B is not ample in general. For the definitions of weakly 1-complete manifold and positive line bundles, see [2]. The author would like to thank Dr. S. Iitaka for valuable criticisms.

1. Let p and q be relatively prime positive integers. Let $V_{p,q}$ be a surface in $\boldsymbol{C}^3$ defined by $z_1^p + z_2^q + z_3^{pq} = 0$. It is shown by Ono and Watanabe [3], that the singularity of $V_{p,q}$ is resolved by a manifold $M_{p,q}$ with exceptional set $S_{p,q}$ which is a non-singular curve of genus $\frac{1}{2}(p-1)(q-1)$ and self-intersection number -1.

We use the following elementary

Lemma. *For any $c>0$ there exists $a_0>c$ such that if $a>a_0$ and $x>a$,*

$$(x-a)^a < (a-c)^c \exp(6(a-c)x - 3(a^2-c^2)).$$

Let p_i $(i=1, 2, \cdots)$ be an increasing sequence of prime integers greater than 3 such that every p_{i+1} satisfies the property of a_0 if we take p_i for c in the lemma. This condition implies that any two of the functions

$$x_1^3 \exp(-3(x_2-p_i)^2) + (x_2-p_i)^{p_i}, \qquad i=1, 2, \cdots$$

have no common zeroes on $\boldsymbol{R}^2$. Let V' be a real analytic subvariety of $\boldsymbol{R}^3$ defined by

$$x_3 = f(x_1, x_2) := \sum_{p \in A} \exp(-x_1^2 - (x_2-p)^2) \times (x_1^3 \exp(-3(x_2-p)^2) + (x_2-p)^p)^{1/3p}$$

where $A = \{p_i ; i=1, 2, \cdots\}$. Note that at $(u_1, u_2) \in \boldsymbol{R}^2$ either $f(x_1, x_2)$ is real analytic or $f(x_1, x_2) - \exp(-x_1^2 - (x_2-p)^2)(x_1^3 \exp(-3(x_2-p)^2) + (x_2-p)^p)^{1/3p}$ is real analytic where $u_1^3 \exp(-3(u_2-p)^2) + (u_2-p)^p = 0$. Hence V' is well defined.

We apply a theorem of Grauert ([1, Proposition 7]) and get a connected complex hypersurface V of a Stein neighbourhood of $\boldsymbol{R}^3$ in $\boldsymbol{C}^3$

such that $V\cap \boldsymbol{R}^3=V'$ and Sing $V=\{(0, \mathrm{p}, f(0, p)) ; p \in A\}$ where Sing V denotes the singular locus of V. Let $\pi : X \to V$ be a resolution of V such that every singular point $(0, p, f(0, \mathrm{p}))$ is resolved by a non-singular curve of genus $p-1$ and self-intersection number -1. Since V is Stein, X is weakly 1-comlete. Clearly the line bundle $\left[-\sum_{u \in \mathrm{Sing} V} \pi^{-1}(u)\right]$ over X is positive but not ample.

One may obtain in a more complicated manner an example of the same sort by constructing a connected hypersurface in $\boldsymbol{C}^3$ with prescribed singularities.

References

[1] Grauert, H.: On Levi's problem and the imbedding of real-analytic manifolds. Ann. of Math., **68**, 460–472 (1958).
[2] Nakano, S.: On the inverse of monoidal transformation. Publ. RIMS, Kyoto University, **6**, 483–502 (1970/71).
[3] Ono, I., and Watanabe, K.: On the singularity of $z^p+y^q+x^{pq}=0$. Sci. Rep. Tokyo Kyoiku Daigaku, Sec. A, **12**, 123–128 (1974).