67. *Convergence and Approximation of Integral Solutions of Nonlinear Evolution Equations*

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1. Introduction. Let X be a Banach space with norm $|\ |$. A subset A of $X\times X$ is said to be ω-accretive if $\tau(x_1-x_2, y_1-y_2)+\omega|x_1-x_2|$ is non-negative for every $[x_i, y_i]\in A$, $i=1,2$, where $\tau(x,y)=\inf_{\lambda>0}\lambda^{-1}(|x+\lambda y|-|x|)$ for $x, y\in X$.

Consider the following Cauchy problem

(1) $$du/dt+Au\ni 0,\quad 0\leqq t<T,\quad u(0)=x.$$

According to Bénilan [1], a continuous function $u:[0,T)\to X$ is called an integral solution of type ω (for simplicity an ω-integral solution) of (1), if it satisfies $u(0)=x$ and

(2) $$e^{-\omega t}|u(t)-u|-e^{-\omega s}|u(s)-u|\leqq\int_s^t e^{-\omega\sigma}\,\tau(u(\sigma)-u,\ -v)d\sigma$$

for every $[u,v]\in A$ and $0\leqq s\leqq t<T$.

Concerning the existence of an ω-integral solution of (1) in a general Banach space, sufficient conditions were given by Crandall and Liggett [4], Bénilan [1], Y. Kobayashi [5] and Pierre [8]. Some of them were then applied by Brezis and Pazy [3], Kurtz [6], Miyadera and Kobayashi [7] and others to obtain convergence and approximation theorems for the semigroups corresponding to ω-integral solutions.

In this paper we deal with some problems of similar nature, but in a slightly different manner. Our method does not depend upon any theorem on generation of nonlinear semigroups. Instead, we make use of a necessary condition for an X-valued function to be an ω-integral solution of (1) (Lemma 1). Assuming the existence of an ω-integral solution $u(t)$ of (1), we estimate the error of it, the difference between $u(t)$ and its approximation throughout. Our results, the statements of which appear somewhat complicated, still include most of the results obtained by the previous authors.

2. The main theorems. Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of subsets $\subset X\times X$. By $\mathrm{Lim}\,A_n\supset A$ we mean that A_n converge to A in the sense of Kurtz [6], that is, for every $[u,v]\in A$ there exist $[u_n, v_n]\in A_n$ such that $\lim(|u_n-u|+|v_n-v|)=0$.

We first study a relation between the convergence of ω_n-integral solutions $u_n(t)$ of the Cauchy problems

$(1)_n$ $$du/dt+A_nu\ni 0,\quad 0\leqq t<T,\quad u(0)=x_n$$

and the convergence of A_n and x_n.

Theorem 1. *Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of ω_n-accretive sets $\subset X\times X$. Suppose that* $\mathrm{Lim}\, A_n \supset A$, $\lim x_n = x$ *and* $\omega_n \leqq \omega$, $n=1,2,\cdots$.

If ω_n-integral solutions $u_n(t)$ of $(1)_n$ and an ω-integral solution $u(t)$ of (1) exist, then it holds that

$$\limsup |u_n(t)-u(t)| \leqq 2e^{\omega t}|x-u| + 2\prod_{k=1}^{m}(1-\lambda_k\omega_0)^{-1}\Big\{|x_0-u| + \sum_{k=1}^{m}|x_k+\lambda_k y_k - x_{k-1}| + e^{\omega_0 t}\Big(\Big(t-\sum_{k=1}^{m}\lambda_k\Big)^2 + \lambda t\Big)^{1/2}|v|\Big\}$$

for any $x_0\in X$, $\{[x_k, y_k]\}_{k=1}^{m}\subset A$, $\{\lambda_k\}_{k=1}^{m}\subset(0, 1/\omega_0)$; $[u, v]\in A$ and $t\in[0,T)$, where $\omega_0=\mathrm{Max}(0,\omega)$ and $\lambda=\mathrm{Max}_{1\leqq k\leqq m}\lambda_k$.

Remark. This theorem implies that $u_n(t)$ converge to $u(t)$ uniformly on $[0, T)$ if x belongs to $\overline{D(A)}$ and A generates a semigroup in the sense of [4] or [5]. In [3] and [7] not only A but A_n were assumed to generate semigroups.

Our second theorem extends a result in [7].

Theorem 2. *Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of ω_n-accretive sets $\subset X\times X$ and $\{U_n\}_{n=1}^{\infty}$ be a sequence of lipschitz operators: $\overline{D(A_n)}\to X$ with constants $M_n\geqq 1$ satisfying $R(I+h_nA_n)\supset U_n\overline{D(A_n)}$ for a $\{h_n\}_{n=1}^{\infty}$ such that $h_n\downarrow 0$. Suppose that* $\mathrm{Lim}\,(A_n+(I-U_n)/h_n)\supset A$, $\overline{D(A_n)}\ni x_n\to x$ *and* $\mathrm{Max}\,(0,\omega_n)+(M_n-1)/h_n\leqq\omega$, $n=1,2,\cdots$. *If an ω-integral solution $u(t)$ of (1) exists, then*

$$\limsup |u(t)-\{(I+h_nA_n)^{-1}U_n\}^{[t/h_n]}x_n| \leqq 2e^{\omega t}|x-u| + 2\prod_{k=1}^{m}(1-\lambda_k\omega)^{-1}\Big\{|x_0-u| + \sum_{k=1}^{m}|x_k+\lambda_k y_k - x_{k-1}| + e^{\omega t}\Big(\Big(t-\sum_{k=1}^{m}\lambda_k\Big)^2 + \lambda\sum_{k=1}^{m}\lambda_k + \lambda t\Big)^{1/2}|v|\Big\}$$

holds for any $x_0\in X$, $\{[x_k, y_k]\}_{k=1}^{m}\subset A$, $\{\lambda_k\}_{k=1}^{m}\subset(0,1/\omega)$; $[u,v]\in A$ and $t\in[0,T)$, where $\lambda=\mathrm{Max}_{1\leqq k\leqq m}\lambda_k$.

Corollary. *Let A be an ω-accretive set $\subset X\times X$ and $F(h)$, $h>0$ be a family of contractions: $\overline{D(A)}\to X$ satisfying $R(I+hA)\supset F(h)\overline{D(A)}$ for small $h>0$. Suppose that B is single valued with $D(B)\supset D(A)$ and that $\lim_{h\downarrow 0}h^{-1}(I-F(h))u=Bu$ for every $u\in D(A)$. If $A+B$ generates a semigroup in the sense of Kobayashi for example, then the unique ω-integral solution $e^{-t(A+B)}x$ of $du/dt+(A+B)u\ni 0$, $0\leqq t<\infty$, $u(0)=x\in\overline{D(A)}$ is approximated as*

$$e^{-t(A+B)}x=\lim_{h\downarrow 0}\{(I+hA)^{-1}F(h)\}^{[t/h]}x,$$

where the limit is uniform on any bounded t-intervals $\subset[0,\infty)$.

3. The proof of the results. Here are our key lemmas. Theorem 1 is a direct consequence of Lemma 1, and Theorem 2 follows from Lemmas 1 and 2. The proof is quite similar to that in [7]. Therefore, in this section we confine ourselves to proving the lemmas.

Lemma 1. *Let $A\subset X\times X$ be ω-accretive. If an ω-integral solution $u(t)$ of* (1) *exists, then it satisfies*

$$
\begin{aligned}
|u(t)-x_m| &\leqq e^{\omega t}|x-u|+\prod_{k=1}^{m}(1-\lambda_k\omega_0)^{-1} \\
&\times\left\{|x_0-u|+\sum_{k=1}^{m}|x_k+\lambda_k y_k-x_{k-1}|+e^{\omega_0 t}\left(\left(t-\sum_{k=1}^{m}\lambda_k\right)^2+\lambda t\right)^{1/2}|v|\right\}
\end{aligned}
\tag{3}
$$

for any $x_0\in X$, $\{[x_k, y_k]\}_{k=1}^{m}\subset A$, $\{\lambda_k\}_{k=1}^{m}\subset(0, 1/\omega_0)$; $[u, v]\in A$ *and* $t\in[0, T)$.

Remark. This lemma is a generalization of the approximation theorem by Brezis and Pazy [2]. Moreover, (3) suggests a direct and simple proof of uniqueness and continuous dependance on the initial value of the ω-integral solution of (1) under a certain restriction on A. The well-known Bénilan's method was rather technical and complicated. It is also to be noted that this lemma remains true even if the ω-accretiveness of A is replaced by the condition that $\tau(x_1-x_2, y_1)+|y_2|+\omega|x_1-x_2|$ is non-negative for $[x_i, y_i]\in A$, $i=1, 2$.

Proof. We will show that $u(t)$ satisfies

$$
|u(t)-u|\leqq e^{\omega t}|x-u|+e^{\omega_0 t}t|v|, \tag{4}
$$

$$
\begin{aligned}
e^{-\omega_i t}|u(t)-x_i| &\leqq |x-u|+\sum_{k=1}^{i}(1-\lambda_k\omega_0)^{-1} \\
&\times\left\{|x_0-u|+\sum_{k=1}^{i}\lambda_k|v|+\sum_{k=1}^{i-1}|x_k+\lambda_k y_k-x_{k-1}|+e^{-\omega_i t}|x_i+\lambda_i y_i-x_{i-1}|\right\} \\
&+\frac{1}{\lambda_i}\int_0^t e^{-\omega_i\sigma}|u(\sigma)-x_{i-1}|\,d\sigma, \qquad i=1, \cdots, m,
\end{aligned}
\tag{5}
$$

where $\omega_i=\omega-1/\lambda_i$. If (4) and (5) are true, (3) can easily be proved by induction with the aid of a simple inequality

$$
C+\frac{1}{h}\int_0^t e^{\sigma/h}((\sigma-C+h)^2+\lambda\sigma)^{1/2}d\sigma\leqq e^{t/h}((t-C)^2+\lambda t)^{1/2}, \qquad t\geqq 0
$$

for every h, λ and C with $0<h\leqq C$, $0<h\leqq\lambda$.

Now, (4) is clear from (2). Dealing with (2) again, we obtain

$$
\begin{aligned}
&e^{-\omega t}|u(t)-u|-e^{-\omega s}|u(s)-u| \\
&\quad\leqq -\frac{1}{h}\int_s^t e^{-\omega\sigma}|u(\sigma)-u|\,d\sigma+\frac{1}{h}\int_s^t e^{-\omega\sigma}\tau(u(\sigma)-u, u(\sigma)-u-hv)d\sigma,
\end{aligned}
$$

which is, as is easily verified, equivalent to

$$
\begin{aligned}
&e^{-(\omega-1/h)t}|u(t)-u|-e^{-(\omega-1/h)s}|u(s)-u| \\
&\quad\leqq\frac{1}{h}\int_s^t e^{-(\omega-1/h)\sigma}\tau(u(\sigma)-u, u(\sigma)-u-hv)d\sigma \qquad (h>0).
\end{aligned}
$$

Applying this inequality, we have

$$
\begin{aligned}
e^{-\omega_i t}|u(t)-x_i| &\leqq |x-x_i|+(1-\lambda_i\omega)^{-1}|x_i+\lambda_i y_i-x_{i-1}|(e^{-\omega_i t}-1) \\
&+\frac{1}{\lambda_i}\int_0^t e^{-\omega_i\sigma}|u(\sigma)-x_{i-1}|\,d\sigma.
\end{aligned}
\tag{6}
$$

The ω-accretiveness of A implies $(1-\omega_i\lambda)|x_i-u|\leqq|x_{i-1}-u|+\lambda_i|v|+|x_i+\lambda_i y_i-x_{i-1}|$ and therefore

$$
|x_i-x|\leqq|x-u|+\prod_{k=1}^{i}(1-\lambda_k\omega_0)^{-1}
$$

$$\times\left\{|x_0-u|+\sum_{k=1}^{i}\lambda_k|v|+\sum_{k=1}^{i}|x_k+\lambda_k y_k-x_{i-1}|\right\}. \tag{7}$$

Combining (6) and (7), we have (5). Q.E.D.

Lemma 2. *Let $A\subset X\times X$ be ω-accretive and U be a lipschitz operator: $\overline{D(A)}\to X$ with constant $M\geqq 1$ satisfying $R(I+hA)\supset U\overline{D(A)}$ for $h\in(0,1/\omega_0)$.*

Then it holds that

$$\begin{aligned}|x_m-\{(I+hA)^{-1}U\}^r x|&\leqq(1-h\omega_0)^{-r}M^r|x-u|\\&+\prod_{k=1}^{m}(1-\lambda_k\omega_1)^{-1}\left\{|x_0-u|+\sum_{k=1}^{m}\left|x_k+\lambda_k\left(y_k+\frac{I-U}{h}x_k\right)-x_{k-1}\right|\right.\\&\left.+(1-h\omega_0)^{-r}M^r\left(\left(rh-\sum_{k=1}^{m}\lambda_k\right)^2+\lambda\sum_{k=1}^{m}\lambda_k+rh^2\right)^{1/2}\left|v+\frac{I-U}{h}u\right|\right\},\\&\qquad r=0,1,\cdots\end{aligned}\tag{8}$$

for any $x_0\in X$, $\{[x_k,y_k]\}_{k=1}^m\subset A$, $\{\lambda_k\}_{k=1}^m\subset(0,1/\omega_0)$; $[u,v]\in A$ and $x\in\overline{D(A)}$, where $\omega_1=\mathrm{Max}\,(0,\omega+(M-1)/h)$ and $\lambda=\mathrm{Max}_{1\leqq k\leqq m}\lambda_k$.

Proof. In view of

$$\begin{aligned}\lambda^{-1}(|x_1+\lambda(y_1+((I-U)/h)x_1)-x_2|-|x_1-x_2|)\\ \geqq\tau(x_1-x_2,y_1+((I-U)/h)x_1)\\ \geqq h^{-1}|x_1-x_2|-\tau(x_1-x_2,-y_2-h^{-1}x_2+h^{-1}Ux_1)+\tau(x_1-x_2,y_1-y_2),\end{aligned}$$

we find that

$$(\lambda+h-\omega\lambda h)|x_1-x_2|\leqq\lambda|x_2+hy_2-Ux_1|+h|x_1+\lambda(y_1+((I-U)/h)x_1)-x_2|$$

for every $[x_1,y_1]$ and $[x_2,y_2]\in A$. Making use of this, we obtain

$$\begin{aligned}(\lambda_i+h-\omega\lambda_i h)|x_i-\{I+hA)^{-1}U\}^j x|\\ \leqq\lambda_i M|x_i-\{(I+hA)^{-1}U\}^{j-1}x|+h|x_{i-1}-\{(I+hA)^{-1}U\}^j x|\\ +h|x_i+\lambda_i(y_i+((I-U)/h)x_i)-x_{i-1}|,\end{aligned}\tag{9}$$

$$\begin{aligned}|x_0-\{(I+hA)^{-1}U\}^j x|\leqq(1-h\omega_0)^{-j}M^j|x-u|+|x_0-u|\\ +(1-h\omega_0)^{-j}M^j jh|v+((I-U)/h)u|.\end{aligned}\tag{10}$$

On the other hand, recalling the proof of (7), we have

$$\begin{aligned}|x_i-x|\leqq|x-u|+\prod_{k=1}^{i}(1-\lambda_k\omega_1)^{-1}\left\{|x_0-u|+\sum_{k=1}^{i}\lambda_k\left|v+\frac{I-U}{h}u\right|\right.\\ \left.+\sum_{k=1}^{i}\left|x_k+\lambda_k\left(y_k+\frac{I-U}{h}x_k\right)-x_{k-1}\right|\right\}.\end{aligned}\tag{11}$$

The inequalities (10) and (11) show that (8) is valid for $(m,r)=(0,j),(i,0)$ respectively. By means of (9) the lemma can be proved by induction as in [5]. Q.E.D.

Example due to Kobayashi. Kobayashi informed us, among other things, of the following meaningful example concerning Theorem 1:

Let X be the real line and consider the subsets $A=\{[x,1]:x\geqq 0\}\cup\{[0,0]\}$ and $A_n=\{[x,1]:x>0\}\cup\{[0,1/n]\}$, $n=1,2,\cdots$ of $X\times X$. Clearly A and A_n are accretive (0-accretive) and $\mathrm{Lim}\,A_n\supset A$ holds. The Cauchy problems $(1)_n$ with $u(0)=0$ admit 0-integral solutions $u_n(t)=0$ and $v_n(t)=-t/n$, and A_n fail to generate semigroups of any type ever known.

Nevertheless, both $u_n(t)$ and $v_n(t)$ surely converge to the unique strong solution $u(t)=0$ of (1) with the initial value 0 uniformly on $[0, T)$.

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References

[1] Ph. Bénilan: Equations d'évolution dans un espace de Banach quelconque et application. Thèse Orsay (1972).

[2] H. Brezis and A. Pazy: Accretive sets and differential equations in Banach spaces. Israel J. Math., **8**, 367–383 (1970).

[3] ——: Convergence and approximation of semigroups of nonlinear operators in Banach spaces. J. Func. Anal., **9**, 63–74 (1972).

[4] M. G. Crandall and T. M. Liggett: Generation of semigroups of nonlinear transformations on general Banach spaces. Amer. J. Math., **93**, 265–298 (1971).

[5] Y. Kobayashi: Difference approximation of Cauchy problems for quasi-dissipative operators and generation of nonlinear semigroups. J. Math. Soc. Japan, **27**, 640–665 (1975).

[6] T. G. Kurtz: Convergence of sequences of semigroups of nonlinear operators with an application to Gas kinetics. Trans. Amer. Math. Soc., **186**, 259–272 (1973).

[7] I. Miyadera and Y. Kobayashi: Convergence and Approximation of Nonlinear Semi-Groups. Japan-France Seminar on Functional Analysis and Numerical Analysis (1976).

[8] M. Pierre: Génération et perturbations de semi-groupes de contractions nonlinéaires. Thèse (1976).