7. On the Extension of Analytic Sets

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In this note, we consider the problem of extending analytic sets across a real linear subspace of $\boldsymbol{C}^n$. Concerning the extension of analytic sets across $\boldsymbol{R}^n$, the precise results by B. Shiffman [4], H. Alexander [1] and J. Becker [2] are well known (see also [5]). We note that it is sufficient to consider the case of the extension across $\boldsymbol{C}^m \times \boldsymbol{R}^{n-m}$ in our problem. We call a subset A of an open subset Ω of $\boldsymbol{C}^n$ a thick set if A is not contained in a countable union of locally closed analytic sets of codimension 1 in Ω. For the properties of thick sets, see [5] Appendix of Chap. 2.

Theorem 1. *Suppose Ω_1 is a connected open subset of $\boldsymbol{C}^m$, Ω_2 is an open subset of $\boldsymbol{C}^{n-m}$ and V is a pure $m+1$-dimensional analytic set in $\Omega_1 \times (\Omega_2 - \boldsymbol{R}^{n-m})$. If A is a thick set in Ω_1 such that, for each point p in A, an analytic set $V(p) := V \cap (\{p\} \times \Omega_2)$ in $\{p\} \times (\Omega_2 - \boldsymbol{R}^{n-m})$ can be extended to an analytic set in $\{p\} \times \Omega_2$, then V can be extended to an analytic set in $\Omega_1 \times \Omega_2$.*

This theorem implies the following:

Theorem 2. *Let Ω_1, Ω_2 be the same as the above and let $\Omega = \Omega_1 \times \Omega_2 \subset \boldsymbol{C}^n$. Suppose V is a pure $m+1$-dimensional analytic set in $\Omega - \boldsymbol{C}^m \times \boldsymbol{R}^{n-m}$ such that $\overline{V}$ is analytic in a neighbourhood of a point $(p, q) \in \Omega$. Then $\overline{V}$ is analytic in a neighbourhood of $\Omega_1 \times \{q\}$ in Ω.*

We easily obtain the following:

Corollary 1. *Suppose Ω is an open subset of $\boldsymbol{C}^n$ and V is an analytic set of pure dimension $k \geqq m+2$ in $\Omega - \boldsymbol{C}^m \times \boldsymbol{R}^{n-m}$. Then $\overline{V} \cap \Omega$ is analytic in Ω.*

Particularly, in the case $m=0$, this means Shiffman's result that an analytic set of pure dimension $\geqq 2$ can be extended across $\boldsymbol{R}^n$.

If we combine Theorem 1 with Alexander's result on the extension of the self-conjugate 1-dimensional analytic sets across $\boldsymbol{R}^n$, we obtain the following:

Corollary 2. *Under the assumption of our theorem, we suppose A is a thick set of Ω_1 and for each point p of A, $V(p)$ is self-conjugate in $\Omega_2 - \boldsymbol{R}^{n-m}$, i.e., $\tau V(p) = V(p)$ where $\tau(w^1, \cdots, w^{n-m}) = (\overline{w}^1, \cdots, \overline{w}^{n-m})$ $((w^1, \cdots, w^{n-m}) \in \boldsymbol{C}^{n-m})$. Then $\overline{V} \cap \Omega$ is an analytic set in Ω.*

For the proof of Theorem 1 we need the following basic lemma on the extension of holomorphic functions.

Lemma. *Suppose Ω is a connected open subset of $\boldsymbol{C}^m$, Δ is a unit disc in $\boldsymbol{C}$ with center at the origin and $f(z, w)$ is a bounded holomorphic function on $\Omega \times (\Delta - \boldsymbol{R})$. If A is a thick set in Ω such that, for each point p in A, a holomorphic function $f(p, w)$ of w on $(\Delta - \boldsymbol{R})$ can be extended to a holomorphic function on Δ, then $f(z, w)$ can be extended to a holomorphic function on $\Omega \times \Delta$.*

We shall give the outline of the proof of Theorem 2 according to Becker's method (see [2]). Theorem 1 can be proved by the similar argument as in [2] if we take account of Theorem 2 and the property of thick sets.

We set $D := \{z \in \Omega_1;\ \overline{V} \cap \Omega$ is analytic in a neighbourhood of $(z, q) \in \Omega_1 \times \Omega_2\}$. Then D is non-empty open subset of Ω_1 by our assumption. Since Ω_1 is connected, we have only to show that D is a closed subset of Ω_1, that is, for each boundary point $p \in \partial D \cap \Omega$, $\overline{V} \cap \Omega$ is analytic in a neighbourhood of $(p, q) \in \Omega$. Using a well-known argument, after a suitable linear transformation of coordinates, we may assume (p, q) is the origin $(0, 0)$ of $\boldsymbol{C}^n (= \boldsymbol{C}^n \times \boldsymbol{C}^{n-m})$ and the dimension of $V(0) = V \cap (\{0\} \times \Omega_2)$ is not larger than 1 (see e.g., [5] p. 35, (2.2) Lemma).

According to Lemma 2 of [2], we can take a suitable polynomial function f on $\boldsymbol{C}^{n-m}$ defined by

$$f(z') = \sum_{i=1}^{n-m} a_i z_i'^2, \qquad \text{where } a_i > 0$$

so that the set $\{f=0\} \cap V(0)$ is discrete in $\{0\} \times (\Omega_2 - \boldsymbol{R}^{n-m})$. We define a holomorphic map $g: \boldsymbol{C}^n (= \boldsymbol{C}^m \times \boldsymbol{C}^{n-m}) \to \boldsymbol{C}^{n+1} (= \boldsymbol{C}^m \times \boldsymbol{C} \times \boldsymbol{C}^{n-m})$ by

$$g(z, z') = (z, f(z'), z') \qquad \text{where } z \in \boldsymbol{C}^m,\ z' \in \boldsymbol{C}^{n-m}.$$

We set

$$B = g(\boldsymbol{C}^m \times \boldsymbol{R}^{n-m}) = \{(z, y, x);\ z \in \boldsymbol{C}^m, x \in \boldsymbol{R}^{n-m}, y = f(x)\}$$

and let $\tilde{\Omega} = \Omega_1 \times \boldsymbol{C} \times \Omega_2$.

Then we see easily $g_{|\Omega}: \Omega \to \tilde{\Omega}$ is a proper injective map, so $g(V)$ is a pure $m+1$-dimensional analytic set in $\tilde{\Omega} - B$, $g(\overline{V} \cap \Omega) = \overline{g(V)} \cap \tilde{\Omega}$, and hence $\overline{g(V)}$ is analytic in a neighbourhood of $g(D)$ in $\tilde{\Omega}$. We define the projection map $\pi: \boldsymbol{C}^{n+1} \to \boldsymbol{C}^{m+1}$ by $\pi(z, w, z') = (z, w)$, where $z \in \boldsymbol{C}^m$, $w \in \boldsymbol{C}$, $z' \in \boldsymbol{C}^{n-m}$. Since $\{f=0\} \cap V(0)$ is discrete in $\{0\} \times (\Omega_2 - \boldsymbol{R}^{n-m})$, $\pi^{-1}(0) \cap [\overline{g(V)} \cap \tilde{\Omega}]$ is a countable set.

Then we can take polydiscs $\Delta_1', \Delta_2', \Delta''$ in $\boldsymbol{C}^m, \boldsymbol{C}, \boldsymbol{C}^{n-m}$ respectively with center at the origins which satisfy the conditions

i) $(\Delta_1' \times \Delta_2' \times \partial \Delta'') \cap \overline{g(V)} = \emptyset$, $\Delta_1' \times \Delta_2' \times \Delta'' \subset \tilde{\Omega}$, and

ii) There is a non-empty open subset U of Δ_1' such that $\overline{g(V)}$ is analytic in $U \times \Delta_2' \times \Delta''$.

In fact, according to the above fact, it is easy to take polydiscs $\Delta_1' \subset \boldsymbol{C}^m$, $\Delta_2 \subset \boldsymbol{C}$, $\Delta'' \subset \boldsymbol{C}^{n-m}$ with center at origins satisfying i) only. Let

$\Delta_2(r)$ be a disc in C of radius r with center at the origin which is contained in Δ_2. Since ∂D contains the origin, we can take a non-empty relatively compact open subset U of $D \cap \Delta_1' (\neq \emptyset)$. Let $\tilde{D}$ be a neighbourhood of $D \times (0, 0)$ $(= g(V))$ in which $\overline{g(V)}$ is analytic.

We shall show that $\overline{g(V)}$ is analytic in $U \times \Delta_2(r) \times \Delta''$ for sufficiently small $r > 0$. Let $E(r)$ be the subset of $U \times \Delta_2 \times \Delta''$ in which $\overline{g(V)}$ is not analytic. Then $E(r)$ is contained in $(U \times \Delta_2(r) \times \Delta'') \cap B$ and we can easily see that $(U \times \Delta_2(r) \times \Delta'') \cap B$ is contained in $U \times \{y \in \boldsymbol{R}; 0 \leqq y < r\} \times S(r)$, where $S(r) = \{x \in \boldsymbol{R}^{n-m}; \sum_{i=1}^{n-m} a_i x_i^2 < r\}$ by the definition of B. On the other hand, as U is relatively compact in D, we can easily see that $U \times \{y \in \boldsymbol{R}; 0 \leqq y < r\} \times S(r)$ is contained in $\tilde{D}$ for sufficiently small $r > 0$. Therefore $E(r) = \emptyset$ for sufficiently small $r > 0$. We set $\Delta_2' = \Delta_2(r)$ for such $r > 0$. Then we can see $\Delta_1', \Delta_2', \Delta''$ satisfy i) and ii). By i) and ii) we see that

$$\pi : (\Delta_1' \times \Delta_2' \times \Delta'') \cap \overline{g(V)} \to \Delta_1' \times \Delta_2'$$

is a proper map and hence,

$$\pi : ((\Delta_1' \times (\Delta_2' - \bar{R}_+) \times \Delta'') \cup (U \times \Delta_2' \times \Delta'')) \cap \overline{g(V)} \\ \to (\Delta_1' \times (\Delta_2' - \bar{R}_+)) \cup (U \times \Delta_2'),$$

where $\bar{R}_+ = \{x \in \boldsymbol{R}; x \geqq 0\}$, is an analytic cover (c.f., [3]). We can see that $(\Delta_1' \times (\Delta_2' - \bar{R}_+) \times \Delta'') \cap g(V)$ is dense in $(\Delta_1' \times \Delta_2' \times \Delta'') \cap \overline{g(V)}$, because for every $z_0 \in \Delta_1' \times \Delta_2'$, $\pi^{-1}(z_0) \cap g(V)$ is discrete in $\{z = z_0\} \cap (\Delta_1' \times \Delta_2' \times \Delta'' - B)$. Therefore, by the use of the above lemma, we can show that $\overline{g(V)}$ is analytic in $\Delta_1' \times \Delta_2' \times \Delta'' \subset \boldsymbol{C}^{n+1}$, and hence $\bar{V} \cap \Omega$ is analytic in a neighbourhood of $p = 0 \in \boldsymbol{C}^n$.

References

[1] Alexander, H.: Continuing 1-dimensional analytic sets. Math. Ann., **191**, 143–144 (1971).

[2] Becker, J.: Continuing analytic sets across $\boldsymbol{R}^n$. Math. Ann., **195**, 103–106 (1972).

[3] Gunning, R. C., and H. Rossi: Analytic Function of Several Complex Variables. Prentice-Hall (1965).

[4] Shiffman, B.: On the continuation of analytic sets. Math. Ann., **185**, 1–12 (1970).

[5] Siu, Y.-T.: Techniques of Extension of Analytic Objects. Marcel Dekker. Inc. (1974).