45. *The Structure of Bebutov Dynamical System*

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1. Introduction. Let X be a metric space. A flow or a dynamical system on X is defined to be the triple (X, R, π) consisting of X, the real line R and a map $\pi: X\times R \to X$ such that

a) $\pi(x, 0)=x$, $x\in X$,

b) $\pi(\pi(x, s), t)=\pi(x, s+t)$, $s, t\in R$, $x\in X$,

c) π is continuous on $X\times R$.

Given a dynamical system on X, the space X is called the phase space of the dynamical system.

Let X_u be the set of all complex-valued continuous functions on R. X_u becomes a metric space with the metric

$$\rho(\varphi, \psi)=\sup_{T>0} \min \left\{\max_{|x|\leq T} |\varphi(x)-\psi(x)|, \frac{1}{T}\right\}.$$

Define a map

$$f_u: X_u\times R \longrightarrow X_u$$

by

$$f_u(\varphi, t)=\varphi\circ g_t, \qquad \varphi\in X,\ t\in R,$$

where $g_t(x)=x+t$ for any $x\in R$. Then a dynamical system (X_u, R, f_u) is obtained, which is called the Bebutov dynamical system [1]. The Bebutov dynamical system is important in the sense that a large class of compact flows (i.e., the flows such that the phase spaces are compact) may be embedded in it by virtue of the theorem of Bebutov-Kakutani [2]: a necessary and sufficient condition for a compact flow to be isomorphic to some subsystem of the Bebutov dynamical system is that its set of rest points be homeomorphic to some subset of the real line R.

The purpose of this paper is to study the structure of the phase spaces of the Bebutov dynamical system and its compact subsystem.

The results obtained are:

(a) any orbit which is dense in X_u is positively or negatively Poisson stable (Theorem 3.1),

(b) there exists an orbit which is dense in X_u, positively or negatively Poisson stable, and neither positively nor negatively receding (Theorem 3.2),

(c) the phase space of the compact subsystem of the Bebutov dynamical system (X_u, R, f_u) is a border set in X_u (Theorem 4.1).

2. Definitions and notations. The sets

$$\pi(x, R)=\{\pi(x, t) ; t \in R\},$$
$$\pi(x, R^{+})=\{\pi(x, t) ; t \in R^{+}\}, \qquad R^{+}=[0, +\infty),$$

and

$$\pi(x, R^{-})=\{\pi(x, t) ; t \in R^{-}\}, \qquad R^{-}=(-\infty, 0],$$

are respectively called the orbit, the positive semi-orbit, and the negative semi-orbit through x. The sets

$$L^{+}(x)=\{y \in X ; \text{there exists a sequence } \{t_n\} \subset R \text{ with } t_n \to +\infty \text{ and } \pi(x, t_n) \to y\}$$

and

$$L^{-}(x)=\{y \in X ; \text{there exists a sequence } \{t_n\} \subset R \text{ with } t_n \to -\infty \text{ and } \pi(x, t_n) \to y\}$$

are respectively called the positive limit set and the negative limit set of x.

An orbit $\pi(x, R)$ is said to be positively or negatively Poisson stable whenever, respectively, $x \in L^{+}(x)$ or $x \in L^{-}(x)$. An orbit is said to be Poisson stable if it is both positively and negatively Poisson stable.

An orbit $\pi(x, R)$ is said to be positively asymptotic if $L^{+}(x)$ is not empty but $x \notin L^{+}(x)$. A negatively asymptotic orbit is defined similarly.

An orbit $\pi(x, R)$ is said to be positively or negatively receding whenever, respectively, $L^{+}(x)=\phi$ or $L^{-}(x)=\phi$.

A point $x \in X$ is called a rest point if $x=\pi(x, t)$ for all $t \in R$. A point $x \in X$ is said to be periodic if there exists a real number $T \neq 0$ such that

$$\pi(x, t)=\pi(x, t+T) \qquad \text{for all } t \in R.$$

A set $M \subset X$ is called invariant with respect to (X, R, π) whenever $\pi(x, t) \in M$ for all $x \in M$ and all $t \in R$.

A dynamical system (M, R, α) is called a subsystem of (X, R, π) if M is invariant with respect to (X, R, π) and α is a restriction of π to $M \times R$.

3. The structure of the phase space of the Bebtov dynamical system. It is known that the Bebutov dynamical system has an orbit which is dense in X_u [1].

Theorem 3.1. *If $f_u(\varphi, R)$ is dense in X_u, then $f_u(\varphi, R)$ is positively or negatively Poisson stable.*

Proof. Let ψ be any point in $f_u(\varphi, R)$. ψ is not a rest point of (X_u, R, f_u) by the equality

$$\overline{f_u(\psi, R)}=\overline{f_u(\varphi, R)}=X_u. \tag{1}$$

Hence there exists a positive number r such that

$$f_u(\psi, R^{+}) \setminus U(\psi, r) \neq \phi$$

and

$$f_u(\psi, R^{-}) \setminus U(\psi, r) \neq \phi$$

by virtue of [3, p. 16, Theorem 2.6]. Here $U(\psi, r)$ is the open ball of radius r and center ψ.

Define t_+ and t_- as follows:

$$t_+ = \mathrm{lub}\,\{t\,;\, f_u(\psi, [0, t]) \subset U(\psi, r)\},$$
$$t_- = \mathrm{glb}\,\{t\,;\, f_u(\psi, [t, 0]) \subset U(\psi, r)\},$$

where $f_u(\psi, [a, b]) = \{f_u(\psi, s)\,;\, s \in [a, b]\}$. Clearly

$$f_u(\psi, [t_-, t_+]) \subset \overline{U(\psi, r)}.$$

On the other hand, the set of all periodic points of (X_u, R, f_u) is dense in X_u [1], so that we can choose a sequence of periodic points $\{\alpha_n\}$ which satisfies the following conditions:

1) $\alpha_n \in U(\psi, r)$ for all $n \in N$,

2) $\rho(\alpha_n, \psi) \to 0$ monotone as $n \to +\infty$.

Since φ as well as ψ is not periodic by virtue of (1), we have

$$\alpha_n \in U(\psi, r) \setminus f_u(\psi, [t_-, t_+])$$

for all $n \in N$, so that

$$(\forall n \in N) \exists s_n > 0\,;\, U(\alpha_n, s_n) \subset U(\psi, r) \setminus f_u(\psi, [t_-, t_+]) \text{ and } s_n < \rho(\alpha_n, \psi).$$

Further, (1) implies that $U(\alpha_n, s_n)$ contains a point $\eta_n \in f_u(\psi, R)$ for each $n \in N$. Let $u_n \in R$ be such that $\eta_n = f_u(\psi, u_n)$. Then

$$u_n \overline{\in} [t_-, t_+] \qquad \text{for all } n \in N.$$

Here it follows that

$$\lim_{n \to +\infty} \rho(f_u(\psi, u_n), \psi) = 0 \tag{2}$$

by virtue of the inequality

$$\rho(f_u(\psi, u_n), \psi) = \rho(\eta_n, \psi) \leqq \rho(\eta_n, \alpha_n) + \rho(\alpha_n, \psi) < s_n + \rho(\alpha_n, \psi).$$

However, the sequence $\{u_n\}$ is unbounded. For, if $\{u_n\}$ is bounded, then it has an accumulation point, say v. Let $\{v_n\}$ be a subsequence of $\{u_n\}$ converging to v. Then

$$\rho(f_u(\psi, v), \psi) = 0$$

follows by virtue of (2), so that

$$f_u(\psi, v) = \psi, \tag{3}$$

whereas

$$v \overline{\in} (t_-, t_+),$$

since $v_n \overline{\in} [t_-, t_+]$ for all $n \in N$. Further, $v \neq 0$, because $t_- < 0$ and $0 < t_+$. This fact and (3) imply that ψ is a periodic point of (X_u, R, f_u), which is a contradiction, since ψ is not periodic by virtue of (1).

Thus the sequence $\{u_n\}$ has a subsequence $\{w_n\}$ diverging to $+\infty$ or $-\infty$. If $\{w_n\}$ diverges to $+\infty$, then

$$\rho(f_u(\psi, w_n), \psi) \longrightarrow 0 \qquad (n \to +\infty),$$

so that

$$\psi \in L^+(\psi). \tag{4}$$

Since $\psi \in f_u(\varphi, R)$ by the assumption, it follows from (4) that $\varphi \in L^+(\varphi)$. Thus φ is positively Poisson stable. In case $\{w_n\}$ diverges to $-\infty$, we can show in the same way as above that φ is negatively Poisson stable.

Q.E.D.

Theorem 3.2. *The Bebutov dynamical system* (X_u, R, f_u) *has an orbit which is dense in* X_u*, positively or negatively Poisson stable, and neither positively nor negatively receding.*

For the proof we shall need the following:

Example 3.3 [1]. Let $\{I_{ij};\, i, j \in N\}$ be a family of closed intervals in R such that

a) $\lim_{i\to+\infty} mI_{ij} = +\infty$ for all $j \in N$,

b) $\lim_{j\to+\infty} mI_{ij} = +\infty$ for all $i \in N$,

c) $I_{ij} \cap I_{kl} = \phi$, unless both $i=k$ and $j=l$

hold, where mI_{ij} is the length of I_{ij}. Since X_u is separable [1], it has a countable subset $S=\{p_k;\, k \in N\}$ which is dense in X_u. Let x_{ij} be the coordinate of the midpoint of I_{ij}. Define a map $\varphi: R \to R$ as follows:

1) $\varphi(x) = p_k(x - x_{ki}),\ x \in I_{ki}$,

2) on the spaces between the intervals the map φ is defined by the linear interpolation. Then, the orbit $f_u(\varphi, R)$ is dense in X_u.

The proof of Theorem 3.2. Choose a family of the intervals on the real line R which satisfy the conditions a), b) and c) in Example 3.3 as follows:

$I_{11} \subset R^-$; $I_{12} \subset R^+$;

I_{21} to the left of I_{11}; I_{31} to the right of I_{12};

I_{22} to the left of I_{21}; I_{13} to the right of I_{31};

$\cdots$.

This procedure continues diagonally in the manner shown by the arrows in the following array (Fig. 1) in which the element (i, j), $i, j \in N$, is the suffix of I_{ij}. Notice that R^+ and R^- each contains infinite elements of $\{I_{ik};\, k \in N\}$ and infinite elements of $\{I_{kj};\, k \in N\}$ as well for all $i, j \in N$. Define a map φ as in Example 3.3, using the family $\{I_{ij}\}$ constructed as above. Then the orbit $f_u(\varphi, R)$ is dense in X_u.

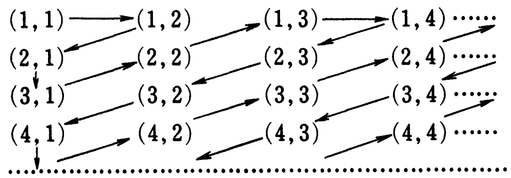

Fig. 1

Now we shall show that $L^+(\varphi)$ and $L^-(\varphi)$ are both non-empty. There exists a subsequence of $S=\{p_k\}$ in Example 3.3 converging to φ, which we denote again by $\{p_k\}$ for simplicity's sake. Then we have

$$(\forall \varepsilon > 0)\exists n_0 \in N;\ (\forall k, l \in N;\ k, l \geqq n_0)\rho(p_k, p_l) \leqq \varepsilon.$$

On the other hand, for any fixed $i \in N$ there exists $n_1 \in N$ such that

$$mI_{ki} > \frac{2}{\varepsilon}, \qquad mI_{li} > \frac{2}{\varepsilon}$$

for every $k, l \in N$ larger than n_1. Let $n^* = \max\{n_0, n_1\}$. Then it follows that for all $k, l \in N$ larger than n^*

$$\rho(p_k, p_l) \leqq \varepsilon, \qquad mI_{ki} > \frac{2}{\varepsilon} \text{ and } mI_{li} > \frac{2}{\varepsilon}.$$

Define $I_{ji} - x_{ji}$ as follows:

$$I_{ji} - x_{ji} = \{x - x_{ji}\,;\, x \in I_{ji}\}, \qquad i, j \in N,$$

where x_{ji} is the midpoint of the interval I_{ji}. Then we have

$$\left[-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}\right] \subset I_{ji} - x_{ji}$$

for every $j \in N$ such that $j \geqq n^*$, which implies that

$$t + x_{ji} \in I_{ji} \qquad \text{for all } t \in \left[-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}\right],$$

so that

$$f_u(\varphi, x_{ji})(t) = p_j(t) \qquad \text{for all } t \in \left[-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}\right].$$

Hence

$$\max_{|t| \leqq 1/\varepsilon} |f_u(\varphi, x_{ji})(t) - p_j(t)| \leqq \varepsilon,$$

which is equivalent to the inequality

$$\rho(f_u(\varphi, x_{ji}), P_j) \leqq \varepsilon, \qquad j \geqq n^*,$$

by virtue of the lemma in [1, p. 420]. Thus we have

$$\begin{aligned} &\rho(f_u(\varphi, x_{li}), f_u(\varphi, x_{ki})) \\ &\quad \leqq \rho(f_u(\varphi, x_{li}), p_l) + \rho(p_l, p_k) + \rho(f_u(\varphi, x_{ki}), p_k) \\ &\quad \leqq \varepsilon + \varepsilon + \varepsilon = 3\varepsilon, \qquad l, k \geqq n^*, \end{aligned}$$

which shows that $\{f_u(\varphi, x_{ji})\,;\, j \in N\}$ is a Cauchy sequence in X_u for each fixed $i \in N$. Consequently this sequence is convergent in X_u for each fixed $i \in N$, since X_u is complete [1]. On the other hand, for each $i \in N$ the sequence $\{x_{ji}\,;\, j \in N\}$ has a subsequence, say $\{y_{ji}\,;\, j \in N\}$, which diverges to $+\infty$. Thus for each $i \in N$ the sequence $\{f_u(\varphi, y_{ji})\,;\, j \in N\}$ converges while $\{y_{ji}\,;\, j \in N\}$ diverges to $+\infty$. This proves $L^+(\varphi) \neq \phi$. The proof of $L^-(\varphi) \neq \phi$ is analogous. Q.E.D.

4. Compact invariant set in the Bebutov dynamical system. The Bebutov-Kakutani theorem suggests us the importance of the study of the compact invariant set in the Bebutov dynamical system.

Theorem 4.1. *The compact invariant sets in the Bebutov dynamical system (X_u, R, f_u) are border sets in X_u.*

Proof. Let K be a compact invariant set in (X_u, R, f_u). K is a proper subset of X_u, since X_u is not compact. Then we have

$$K \cap f_u(\varphi, R) = \phi \tag{1}$$

for every $\varphi \in X_u$ such that $\overline{f_u(\varphi, R)} = X_u$. For, otherwise K will contain $f_u(\varphi, R)$ by virtue of the invariance of K, so that

$$X_u = \overline{f_u(\varphi, R)} \subset K$$

will follow, which is absurd.

Now assume that K contains an interior point, say ψ. ψ has a neighborhood $U(\psi, r) \subset K$. There exists, however, an orbit $f_u(\varphi, R)$ which is dense in X_u (see Example 3.3), so that $U(\psi, r)$ contains a point of $f_u(\varphi, R)$. Hence $K \cap f_u(\varphi, R)$ is not empty, which contradicts the equality (1). Thus the interior of K is empty, i.e., K is a border set.

Q.E.D.

References

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