35. Classification Theory of Non-Complete Algebraic Surfaces

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In this paper we shall show that almost all theorems in classification theory of algebraic surfaces by Enriques, Kodaira, Iitaka, Mumford, Bombieri, etc. can be extended to the case of non-complete algebraic surfaces. We use the following notation:

X: a non-singular algebraic surface (this is the object of the study).

$\bar{X}$: a non-singular complete algebraic surface which contains X as a Zariski open subset.

$D=\bar{X}-X$: the complement of X in $\bar{X}$. We assume that D has only normal crossings.

$\kappa(\bar{X})$ (resp. $\bar{\kappa}(X)$): the Kodaira (resp. logarithmic Kodaira) dimension of $\bar{X}$ (resp. X).

$P_m(\bar{X})$ (resp. $\bar{P}_m(X)$): the m-genus (resp. logarithmic m-genus) of $\bar{X}$ (resp. X) (for the definitions, see [4]).

K: the canonical sheaf of $\bar{X}$.

[ ]: the integral part.

For the sake of simplicity, we shall work only on the ground field $\boldsymbol{C}$, that is, in the case of characteristic zero.

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1. We first construct a relatively minimal model (or a super model) of $(X, \bar{X}, D)$.

Theorem 1. *If $\bar{\kappa}(X)\geqslant 0$, then there exist a non-singular complete surface $\bar{X}_m$, a divisor D_m with coefficients in $\boldsymbol{Q}$ on $\bar{X}_m$ and a birational morphism $f: \bar{X}\to\bar{X}_m$ satisfying the following conditions:*

(1) *$D_m=\sum_i d_iD_i$, $0<d_i\leqslant 1$, where the D_i are irreducible divisors on $\bar{X}_m$.*

(2) *$f^*(K_m+D_m)$ is the arithmetically effective component of $K+D$ in the sense of Zariski* (see Definition 7.6 and Theorem 7.7 of [10]), *where K_m is the canonical sheaf of $\bar{X}_m$.*

$(\bar{X}_m, D_m)$ is obtained by a succession of two kind of steps from $(\bar{X}, D)$ as follows: we denote an intermediate stage by $(\bar{X}', D')$, where $\bar{X}'$ is a non-singular complete algebraic surface and D' is a divisor with coefficients in $\boldsymbol{Q}$ on $\bar{X}'$.

Step 1. If there is an exceptional curve E of first kind on $\bar{X}'$ such that $E\cdot D'\leqslant 1$, then contract E to a point to get a new $\bar{X}'$ and the new D' is the direct image of the old D'.

Step 2. Replace D' by a new D'_1 such that $0\leqslant D'_1\leqslant D'$.

A relatively minimal model $(\bar{X}_m, D_m)$ has a similar nature to a minimal model of complete algebraic surfaces and we can translate Mumford's arguments in [8] to the open case as follows:

Theorem 2. (1) *If $\bar{\kappa}(X)=0$, then $\bar{X}_m$ is a relatively minimal complete surface and there is some integer n such that $[n(K_m+D_m)]=0$.*

(2) *If $\bar{\kappa}(X)=1$, then some high multiple $|n(K_m+D_m)|$ determines a fiber structure $\pi:\bar{X}_m\to C$, which is minimal in the sense of a fiber space. There are two possibilities:*

(2-1) (*Elliptic case.*) *The general fiber of π is an elliptic curve and $D_m=\sum_i F_i$, where the F_i are distinct fibers.*

$$K_m+D_m=\pi^*(K_C+\delta)+\sum_\nu (m_\nu-1)E_\nu+\sum_i F_i,$$

where the $m_\nu E_\nu$ are the multiple fibers and δ is some divisor on C (see [6]). $\dim H^0([n(K_m+D_m)])=n(2g-2+t)+\sum_\nu [n(1-1/m_\nu)]+\sum_i [n/m_i]+1-g$, *for $n\geqslant 2$, where g is the genus of C, $t=\deg\delta$, and the m_i are the multiplicities of the fibers F_i.*

(2-2) (*Quasi-elliptic case.*) *The general fiber of π is a rational curve and $D_m=H+\sum_i d_iF_i$, where H is the horizontal component of D_m and the F_i are fibers of π. The coefficients in H are equal to 1, the degree of H over C is equal to 2, and H has only normal crossings. For every i, $d_i=\frac{1}{2}(1-1/m_i)$ or $(1-1/m_i)$, where m_i is an integer or ∞, if F_i corresponds to a branch point of H or not, respectively. $K_m+D_m=\pi^*(K_C+\delta)+\sum_i d_iF_i$, where $t=\deg\delta=\frac{1}{2}$ (the number of branch points of H). $\dim H^0([n(K_m+D_m)])=n(2g-2+t)+\sum_i [nd_i]+1-g$, for $n\geqslant 2$.*

(3) *If $\bar{\kappa}(X)=2$, then the pluricanonical ring $R=\bigoplus_{n\geqslant 0}H^0(n(K+D))$ is finitely generated. Hence we can define the canonical model $\bar{X}_c$ of $(X,\bar{X},D)$ to be* Proj R. *Then the canonical map $\Phi:\bar{X}\to\bar{X}_c$ is a morphism. Denote by D_c the direct image Φ_*D of D (note that this is only a Weil divisor). Then $\bar{X}_c$ has at most rational singularities or minimal elliptic singularities* (see [7]), *$\bar{X}_m$ coincides with the minimal resolution of $\bar{X}_c$, which factors through Φ, and D_m is determined uniquely by the following conditions:*

(a) *the direct image of* D_m *on* $\bar{X}_c$ *is equal to* D_c,

(b) *for any curve* E_μ *on* $\bar{X}_m$ *which contracts to a point on* $\bar{X}_c$, $(K_m+D_m)E_\mu=0$.

Let E *be the union of those* E_μ. *Then* $E\cup \operatorname{Supp} D_m$ *has only normal crossings. In this case,* $(K_m+D_m)^2>0$, *and* $\bar{P}_2\neq 0$.

Remark. (1) In (1), various examples indicate that $n=12$ is perhaps sufficient.

(2) Note that the uniqueness of $(\bar{X}_m, D_m)$ holds only in case $\bar{\kappa}=2$.

(3) There is no fixed n for all surfaces of $\bar{\kappa}=2$ such that the natural map $\bar{X}_c\to\bar{X}_c^{[n]}$ is an isomorphism, where the latter is the n-canonical model (see [2]). However, for some class of surfaces whose boundaries D satisfy some special hypothesis, $n=6$ is sufficient ([9]).

(4) Note that $\bar{P}_n(X)=\dim H^0([n(K_m+D_m)])$.

2. Next we generalize the theorem of Iitaka [3].

Theorem 3. *The logarithmic Kodaira dimension of algebraic surfaces is invariant under deformations in the sense of* [5]. *Moreover, if* $\bar{\kappa}\neq 2$, *then the logarithmic plurigenera* $\bar{P}_m$ *are also invariant for every* m.

(In case $\bar{\kappa}=2$, the author does not know yet whether $\bar{P}_m$ is invariant or not.)

We prove the theorem by showing the invariance of under global deformations in cases $\bar{\kappa}=-\infty$, 0 and 2.

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