29. *Kodaira Vanishing Theorem and Chern Classes for ∂-Manifolds*

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In the former part of this article, we shall formulate a certain kind of vanishing theorem, which may be a generalization of Kodaira vanishing theorem. In the latter part, we shall define Chern classes of any pair consisting of a compact complex manifold X and a divisor with simple normal crossings. Such a pair (X, D) may be called a ∂-*manifold*. *The vanishing theorem formulated here is a Kodaira* vanishing theorem for ∂-manifolds.

§1. **Theorem 1.** *Let (X, D) be a ∂-manifold and L an ample invertible sheaf on X. Then*

i) $H^q(X, \Omega^p(\log D)\otimes L)=0$, *for* $p+q\geq n+1$,

ii) $H^q(X, \Omega^p(\log D)\otimes L^{-1})=0$, *for* $p+q\leq n$.

Here $n=\dim X$ and $\Omega^p(\log D)$ denotes the sheaf of logarithmic p-forms of X ([1], p. 31).

Proof. Let $D=\sum D_j$ be the decomposition of D into irreducible components and put

$D^k=\{\coprod D_J\,;\, J\subset\{1,\cdots,r\}$, ${}^{\#}J=k$, the D_J denote the intersection of D_j where $j\in J\}$. For example, $D=X$, $D^1=\coprod D_j$.

Following Deligne ([1], p. 32), we define the filtration of $\Omega^p(\log D)$ by $W_k^p=\Omega^k(\log D)\wedge\Omega^{p-k}$. Then we have the following exact sequence

$$0\to W_{k-1}^p\to W_k^p\to \Omega_D^{p-k}\to 0.$$

After making tensor product with L, we obtain the following exact sequence of cohomology groups:

$$\cdots\to H^q(X, W_{k-1}^p\otimes L)\to H^q(X, W_k^p\otimes L)\to H^q(D^k, \Omega_D^{p-k}(L\,|\,D))\to\cdots.$$

Note that $L\,|\,D^k$ is ample, since L is ample. Hence in view of Kodaira vanishing theorem, we have $H^q(D^k, \Omega_D^{p-k}(L\,|\,D^k))=0$, for $p+q\geq n+1$. Hence i) follows. The second assertion ii) follows similarly. Q.E.D.

In order to apply the vanishing theorem, we will use an exact sequence in [4]. Let (X, D) be a ∂-manifold and S an non-singular subvariety of codimention 1 of X, such that $S+D$ has only simple normal crossings. As in [4], we have the map $r_1: \Omega_X^p(\log D)\to\Omega_S^p(\log D\,|\,S)$, and define $\Omega'^p(\log D)=\ker r_1$. Then it follows that $0\to\Omega_X^p(\log D)\to\Omega'^p(\log D)\otimes[S]\to\Omega_S^{p-1}(\log D\,|\,S)\to 0$. Using the sequence above, we derive the following Lefschetz type theorem.

Theorem 2. *The homomorphism induced by the injection $S\subset X$*

$$H^q(X, \Omega_X^p(\log D)) \to H^q(S, \Omega^p(\log D \mid S))$$

is isomorphic for $p+q<\dim X-1$, *and is injective for* $p+q=\dim X-1$.

Corollary. *If* X *is Kähler,* $H^q(X^*, \boldsymbol{C}) \to H^q(S^*, \boldsymbol{C})$ *is isomorphic for* $q<\dim X-1$, *injective for* $q=\dim X-1$, *where* $X^*=X-D$ *and* $S^*=S-D$.

§ 2. Let (X, D) be an n-dimensional ∂-manifold and $T(\log D)$ the dual sheaf of $\Omega^1(\log D)$. Define $c(X, D)=c(T(\log D)) \in H^{2*}(X, \boldsymbol{Z}) =\sum H^{2i}(X, \boldsymbol{Z})$.

Proposition 1. $c(X, D)=c(X) \prod (1+[D_i])^{-1}$.

A proof follows easily from the exact sequence

$$0 \to T(\log D) \to T_X \to N_D \to 0.$$

Theorem 3. $c_n(X, D)[X]=\chi(X^*)$, *which is the Euler characteristic of* $X^*=X-D$.

Proof. In view of the Hodge spectral sequence

$$E_1^{pq}=H^q(X, \Omega^p(\log D)) \Rightarrow H^n(X^*, \boldsymbol{C}),$$

we get

$$\chi(X^*)=\sum(-1)^p \chi(X, \Omega^p(\log D))$$

in which $\chi(X, E)=\sum(-1)^i \dim H^i(X, E) \sharp E$ being a sheaf of holomorphic vector bundle. Recall the Hirzebruch formula of Riemann Roch theorem, $\chi(X, E)=(\sum e^{\delta_i} \prod(\gamma_j/1-e^{-\gamma_j}))[X]$ where $c(T_X)=\prod(1+\gamma_j)$, $c(E)=\prod(1+\delta_i)$. Now we set $E=\Omega^i(\log D)$. Then $c(\Omega^1(\log D))=\prod(1+\delta_i)$. Hence $\chi(X, \Omega^p(\log D))=(\sum \exp(\delta_{i(1)}+\cdots+\delta_{i(p)}) \prod(\gamma_j/1-e^{-\gamma_j}))[X]$. Therefore,

$$\begin{aligned}
\chi(X^*) &=\sum(-1)^p \chi(X, \Omega^p(\log D)) \\
&=(1-e^{\delta_1}) \cdots(1-e^{\delta_n}) \prod(\gamma_j/1-e^{-\gamma_j})[X] \\
&=(-\delta_1) \cdots(-\delta_n) \prod(\gamma_j/1-e^{-\gamma_j})[X] \\
&=(-1)^n c_n(\Omega^1(\log D))[X] \\
&=c_n(T(\log D))[X].
\end{aligned}$$

Q.E.D.

This theorem was suggested by Iitaka, and Iitaka [2] and Kawamata [3] gave different proofs from ours.

Example. $c(D_I, D_J)=(1+h)^{N+1} \prod_{j \in J}(1+a_j h)^{-1}$, where $\sum D_j$ is a divisor on $\boldsymbol{P}^N$ with simple normal crossings, $I \subset J$ are multi-indices $\deg D_j=a_j$ and $h=$the restriction of the fundamental class of $\boldsymbol{P}^N$.

References

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