60. *On the Least Positive Eigenvalue of the Laplacian for Riemannian Manifolds*

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§ 1. **Preliminaries.** Let M be an n-dimensional compact connected manifold. For a Riemannian metric g, let $-\Delta_g$ be the Laplacian associated to g acting on smooth functions on M. We may use the convention that the set of non-zero eigenvalues of Δ_g consists of the eigenvalues repeated a number of time equal to their multiplicities. For a fixed positive integer k, let $\lambda_1(g), \cdots, \lambda_k(g)$ be k eigenvalues chosen as small as possible. We consider the function Ξ_k on the space of smooth Riemannian metrics on M (cf. [1] p. 143):

$$\Xi_k(g)=V_g^{-2/n}\sum_{i=1}^k \lambda_i(g)^{-1},$$

where V_g is the volume of (M, g). For a fixed Riemannian metric g_0, let $m(g_0)$ be the multiplicity of the least positive eigenvalue $\lambda_1(g_0)$ of Δ_{g_0}. The function $\Xi=\Xi_{m(g_0)}$ is called (cf. [1]) to be critical at g_0 if

$$\left[\frac{d}{dt}\Xi(g(t))\right]_{t=0}=0,$$

for every one-parameter family of Riemannian metrics $g(t)$, $g(0)=g_0$, $|t|<\varepsilon$, depending real analytically on t.

§ 2. **Statements of Results.** Let K be a compact connected Lie group, K_0 a closed subgroup of K and $M=K/K_0$ the quotient manifold. Let g_0 be a K-invariant Riemannian metric on M. Then we have the following results:

Theorem 1. *Let $M=K/K_0$ be as above. Suppose that the linear isotropy representation of K_0 is irreducible over $\mathbf{R}$. Then the function Ξ is critical at the K-invariant metric g_0.*

Theorem 2. *Let $M=K/K_0$ be the compact homogeneous space of* $\dim. M\geq 2$. *In case of* $\dim. M>2$, *we assume the linear isotropy representation of K_0 is irreducible over $\mathbf{R}$. Let g_0 be a K-invariant metric on M. Then*

$$\Xi(\varphi g_0)\geq m(g_0)^{2/n-1}\Xi(g_0),$$

for every positive valued smooth function φ on M such that $\langle\varphi^{n/2}, \eta\rangle_{g_0}=0$ for every $\eta\in\mathcal{F}$. Here $\langle\cdot,\cdot\rangle_{g_0}$ is the L_2-inner product on the space of smooth functions on M and $\mathcal{F}$ is the $\lambda_1(g_0)$ eigenspace of Δ_{g_0}.

Remark 1. The function $\varphi^{n/2}$ in Theorem 2 is given as follows, for example: Let ψ be a smooth function orthogonal to $\mathcal{F}$ with respect to $\langle\cdot,\cdot\rangle_{g_0}$. Then $\psi+c$, $(c>\max.|\psi|)$ is a positive valued smooth function orthogonal to $\mathcal{F}$ with respect to $\langle\cdot,\cdot\rangle_{g_0}$. We may put $\varphi^{n/2}=\psi+c$, $(c>\max.|\psi|)$.

Remark 2. These theorems have been obtained by M. Berger [1] in case of $M=S^n$.

§3. Proof of Theorem 1. Let $\{\varphi_i\}_{i=1}^{m(g_0)}$ be an orthonormal base of $\mathcal{F}$ with respect to $\langle\cdot,\cdot\rangle_{g_0}$. It may be proved that they satisfy the Conditions 1 and 2 of Proposition 4.24 in [1] p. 143.

Condition 1. For $k\in K$, the translations $\{k\varphi_i\}_{i=1}^{m(g_0)}$ by k is also an orthonormal basis of $\mathcal{F}$. Then $\sum_{i=1}^{m(g_0)}(k\varphi_i)^2=\sum_{i=1}^{m(g_0)}\varphi_i^2$. Then the sum $\sum_{i=1}^{m(g_0)}\varphi_i^2$ is a constant function C on M by the homogenuity of K on M. Integrating over M, we have

$$CV_0=\langle\textstyle\sum_{i=1}^{m(g_0)}\varphi_i^2,1\rangle_{g_0}=\sum_{i=1}^{m(g_0)}\|\varphi_i\|_{g_0}^2=m(g_0).$$

Here V_0 is the volume of (M,g_0).

Condition 2. Since the isotropy representation of K_0 is irreducible, there exists a constant C' such that

$$\textstyle\sum_{i=1}^{m(g_0)}d\varphi_i\circ d\varphi_i=C'g_0.$$

Then we have

$$\int\mathrm{trace}_{g_0}(C'g_0)v_{g_0}=C'n=\int\mathrm{trace}_{g_0}(\textstyle\sum_{i=1}^{m(g_0)}d\varphi_i\circ d\varphi_i)v_{g_0}=\sum_{i=1}^{m(g_0)}\|d\varphi_i\|_{g_0}^2,$$

$$\textstyle\sum_{i=1}^{m(g_0)}\|d\varphi_i\|_{g_0}^2=\sum_{i=1}^{m(g_0)}\langle\varDelta_{g_0}\varphi_i,\varphi_i\rangle_{g_0}=\lambda_1(g_0)m(g_0),$$

hence $C'=m(g_0)\lambda_1(g_0)n^{-1}V_0^{-1}$. Q.E.D.

§4. Proof of Theorem 2. Let $g=\varphi g_0$. Then

$$\langle 1,\eta\rangle_g=\langle\varphi^{n/2},\eta\rangle_{g_0}=0\qquad(\text{for every }\eta\in\mathcal{F}).$$

Two inner products $\langle\cdot,\cdot\rangle_{g_0}$, $\langle d\cdot,d\cdot\rangle_g$ can be defined on $\mathcal{F}$. There exists an orthonormal basis $\{\eta_i\}_{i=1}^{m(g_0)}$ of $\mathcal{F}$ with respect to $\langle\cdot,\cdot\rangle_{g_0}$ such that $\langle d\eta_i,d\eta_j\rangle_g=0$ $(i\neq j)$. Then under the assumption that $\langle 1,\eta\rangle_g=0$ $(\eta\in\mathcal{F})$, the following inequality holds (cf. Hersch [3]):

$$(1)\qquad \textstyle\sum_{i=1}^{m(g_0)}\lambda_i(g)^{-1}\geqq\sum_{i=1}^{m(g_0)}\|\eta_i\|_g^2/\|d\eta_i\|_g^2,$$

for an orthogonal basis $\{\eta_i\}_{i=1}^{m(g_0)}$ of $\mathcal{F}$ with respect to $\langle d\cdot,d\cdot\rangle_g$. By means of the choice of $\{\eta_i\}_{i=1}^{m(g_0)}$, we have $\sum_{i=1}^{m(g_0)}\eta_i^2=m(g_0)V_0^{-1}$ (cf. Proof of Theorem 1). Then

$$(2)\qquad \textstyle\sum_{i=1}^{m(g_0)}\|\eta_i\|_g^2=\sum_{i=1}^{m(g_0)}\int_M\eta_i^2v_g=m(g_0)V_0^{-1}V_g,$$

where v_g is the canonical measure associated to the metric g (cf. [2] p. 11).

In case of $n=\dim.M=2$, then

$$\|d\eta_i\|_g^2=\|d\eta_i\|_{g_0}^2=\langle\varDelta_{g_0}\eta_i,\eta_i\rangle_{g_0}=\lambda_1(g_0),$$

by means of $|d\eta|_g=\varphi^{-1/2}|d\eta|_{g_0}$ where $|d\eta|_g$ is the pointwise norm of 1-form $d\eta$ with respect to the metric g, and $\int_M\eta v_g=\int_M\varphi\eta v_{g_0}(\eta\in C^\infty(M))$. Hence, together with (1) and (2), we have

$$\textstyle\sum_{i=1}^{m(g_0)}\lambda_i(g)^{-1}\geqq m(g_0)\lambda_1(g_0)^{-1}V_0^{-1}V_g,\quad\text{that is}\quad \varXi(g)\geqq\varXi(g_0).$$

In case of dim. $M>2$, we assume the linear isotropy representation of K_0 is irreducible over $\boldsymbol{R}$. Then

$$\sum_{i=1}^{m(g_0)} d\eta_i \circ d\eta_i = \lambda_1(g_0)m(g_0)n^{-1}V_0^{-1}g_0$$

(cf. Proof of Theorem 1), hence

$$|d\eta_i|_{g_0}^2 \leqq \sum_{i=1}^{m(g_0)} |d\eta_i|_{g_0}^2 = \mathrm{trace}_{g_0}\left(\sum_{i=1}^{m(g_0)} d\eta_i \circ d\eta_i\right) = \lambda_1(g_0)m(g_0)V_0^{-1}.$$

Therefore we have

$$(3) \qquad \int_M |d\eta_i|_{g_0}^n v_{g_0} \leqq \left(\int_M |d\eta_i|_{g_0}^2 v_{g_0}\right) m(g_0)^{(n-2)/2}\lambda_1(g_0)^{(n-2)/2}V_0^{-(n-2)/2} = m(g_0)^{(n-2)/2}\lambda_1(g_0)^{n/2}V_0^{-(n-2)/2},$$

by $\int_M |d\eta_i|_{g_0}^2 v_{g_0} = \lambda_1(g_0)$. On the other hand,

$$(4) \qquad \|d\eta_i\|_g^2 = \int_M |d\eta_i|_g^2 v_g \leq \left(\int_M (|d\eta_i|_g^2)^{n/2} v_g\right)^{2/n}\left(\int_M v_g\right)^{(n-2)/n} = V_g^{(n-2)/n}\left(\int_M |d\eta_i|_g^n v_g\right)^{2/n}.$$

But, since $|d\eta|_g = \varphi^{-1/2}|d\eta|_{g_0}$ and $\int_M \eta v_g = \int_M \eta\varphi^{n/2}v_{g_0} (\eta \in C^\infty(M))$, we have

$$(5) \qquad \int_M |d\eta_i|_g^n v_g = \int_M |d\eta_i|_{g_0}^n v_{g_0}.$$

Together with (3), (4) and (5), we have

$$\|d\eta_i\|_g^2 \leqq m(g_0)^{(n-2)/n}\lambda_1(g_0)V_0^{-(n-2)/n}V_g^{(n-2)/n}.$$

Therefore from (1) and (2), we have

$$\sum_{i=1}^{m(g_0)} \lambda_i(g)^{-1} \geqq m(g_0)^{2/n}\lambda_1(g_0)^{-1}V_0^{-2/n}V_g^{2/n},$$

that is,

$$\Xi(g) \geqq m(g_0)^{(2/n-1)}\Xi(g_0). \qquad \text{Q.E.D.}$$

References

[1] Berger, M.: Sur les premières valeurs propres des variétés riemanniennes. Compos. Math., **26**(2), 129–149 (1973).

[2] Berger, M., Gauduchon, P., and Mazet, E.: Le spectre d'une variété riemannienne. Springer, Lecture note, **194** (1971).

[3] Hersch, J.: Caractérisation variationnelle d'une somme de valeurs propres consécutives. C. R. A. S., **252**, 1714–1716 (1961).