40. Invariance of Cohomology Groups under a Deformation of an Elliptic System of Linear Differential Equations

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M. Sato has recently proposed to study a "deformation" of systems of linear differential equations in connection with some physical problems (Sato *et al.* [5], Schlesinger [6]). The purpose of this note is to show a theorem concerning the invariance of cohomology groups under the deformation of a system in the sense specified in the theorem below.[1] All equations considered in this note are supposed to be with analytic coefficients.

Theorem. *Let M be a compact real analytic manifold. Let $\mathcal{M}(t)$ be an elliptic system of linear differential equations defined on M with a parameter t running over an open interval I of $\boldsymbol{R}$. Assume that $\mathcal{M}(t)$ is deformed with respect to t in the following sense:*

There exists a system $\mathcal{N}$ of linear differential equations defined on $I\times M$ such that $\{t=t_0\}\times M$ is non-characteristic with respect to $\mathcal{N}$ and that its tangential system $\mathcal{N}_{t_0}$ induced on $\{t=t_0\}\times M$ coincides with $\mathcal{M}(t_0)$ for any t_0 in I. Then

(1) $$\mathrm{Ext}^j(M;\mathcal{M}(t),\mathcal{B}_M)\cong \mathrm{Ext}^j(M;\mathcal{M}(t'),\mathcal{B}_M)$$

holds for any $t,t'\in I$ and any j. Here $\mathcal{B}_M$ denotes the sheaf of hyperfunctions on M.

Proof. Since $\mathcal{N}_t=\mathcal{M}(t)$ is elliptic, $\mathcal{N}$ itself is elliptic. Hence a result of Kawai [3] claims that

(2) $$\mathrm{Ext}^j(I'\times M;\mathcal{N},\mathcal{B}_{\boldsymbol{R}\times M})\cong \mathrm{Ext}^j(I''\times M;\mathcal{N},\mathcal{B}_{\boldsymbol{R}\times M})$$

holds for any j and any open intervals I' and I'' with $I'\subset I''\subset \boldsymbol{R}$. Actually they are known to be finite-dimensional vector spaces over $\boldsymbol{C}$. (See e.g. Guillemin [1], Kawai [3], Kuranishi [4] and references cited there.) Note also that the ellipticity of $\mathcal{N}$ entails that

(3) $$\mathrm{Ext}^j(I'\times M;\mathcal{N},\mathcal{B}_{\boldsymbol{R}\times M})\cong \mathrm{Ext}^j(I'\times M;\mathcal{N},\mathcal{A}_{\boldsymbol{R}\times M})$$

holds for any j and any open set $I'\subset I$. Here $\mathcal{A}_{\boldsymbol{R}\times M}$ denotes the sheaf of real analytic functions on $\boldsymbol{R}\times M$.

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1) The "deformation" considered below is more restricted than that proposed by Sato in that Sato seems to try to include systems with regular singularities to his eventual theory, while apparently he wants to restrict the type of $\mathcal{N}$ so that more precise results could be obtained.

On the other hand, Cauchy-Kowalevsky's theorem (see e.g. Kashiwara [2]) claims that

$$(4)\qquad \varinjlim_{t \in I} \mathrm{Ext}^j (I\times M; \mathcal{N}, \mathcal{A}_{R\times M}) \cong \mathrm{Ext}^j (M; \mathcal{M}(t), \mathcal{A}_M)$$

holds for any j since M is compact. Furthermore, by making use of the ellipticity of $\mathcal{M}(t)$, we conclude that the right hand side of (4) is isomorphic to $\mathrm{Ext}^j (M; \mathcal{M}(t), \mathcal{B}_M)$. Therefore, combining (2), (3) and (4), we obtain the required result. Q.E.D.

Remark 1. One can easily generalize the theorem to the case where there are several parameters $(t_1, \cdots, t_n)$.

Remark 2. If the assumption of the ellipticity of $\mathcal{M}(t)$ is omitted, even the (micro-)local structure of $\mathcal{M}(t)$ is not preserved under the deformation in the sense described above. This topic will be discussed elesewhere.

References

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