36. A Complex Analogue of the Generalized Minkowski Problem

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1. Recently, A. V. Pogorelov [5, 6] announced to have solved the generalized Minkowski problem using the idea of E. Calabi, as was also mentioned in our lecture [3]. It was a key point of solving this problem to reduce it to finding solutions of certain non-linear elliptic partial differential equations defined over the unit sphers S^n $(n \geq 2)$, which we called in [3] as *of the generalized Monge-Ampère type*. In the present note we will show that the framework of finding solutions of the differential equation mentioned above can be applied analogously also in the case of n-complex projective space P_C^n $(n \geq 1)$, instead of the unit sphere. To describe our motivation of studies, we have first to resume and explain the differential equations over S^n appearing in the generalized Minkowski problem which suits to our purpose.

Namely, we denote by ϕ the unknown C^∞-function of n-variables $u_1, u_2, \cdots, u_n$, that is in reality defined over the whole S^n; in fact, if we write the current co-ordinates of the ambient euclidean space $\boldsymbol{R}^{n+1}$ as $(\xi_0, \xi_1, \cdots, \xi_n)$ and cover S^n by the co-ordinates patches $U_i = \{\xi_i \neq 0\}$ $(0 \leq i \leq n)$. In every U_i, we put $u_1 = \xi_0/\xi_i$, $u_2 = \xi_1/\xi_i$, $\cdots$, $u_n = \xi_n/\xi_i$, whereby one considers the differential operator D_i:

$$(1) \qquad D_i(\phi) = |\xi_i|^{-n-2} \det\left(\frac{\partial^2 \phi}{\partial u_j \partial u_k}\right) \qquad (0 \leq i \leq n),$$

then D_i $(0 \leq i \leq n)$ yield the differential operator D defined globally over the sphere S^n. The generalized Minkowski problem for an n-dimensional compact, convex oriented hypersurface V $(n \geq 2)$ is concerned with the following partial differential equation on S^n:

$$(2) \qquad D(\phi) = \kappa,$$

where a given positive function κ on S^n is assumed to satisfy the conditions:

$$\int_{S^n} \kappa \cdot \xi_i dS = 0 \qquad (0 \leq i \leq n),$$

dS denoting the volume element of S^n with respect to the natural metric of S^n (The equation (2) has been known from old times, when $n=2$, as the simplest form of the so-called Monge-Ampère equations [3]). In the

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situation delivered above, one has now to find the *elliptic solutions* ϕ of the equation (2); namely by an elliptic solution ϕ we mean the solution ϕ whose hessian $(\partial^2\phi/\partial u_j\partial u_k)$ is positive definite everywhere on S^n, and A. V. Pogorelov solved this problem. We want to emphasize here that the operator D can be defined intrinsically for arbitrary riemannian manifolds (see [3, 8]). In fact, we can formulate as

$$D(\phi)=\det(\mathrm{Hess}(\phi)+\phi\cdot I_n)$$

where the *symmetric* tensor field, Hess (ϕ), of type (1.1) is given by Hess $(\phi)\cdot X=\nabla_X(\mathrm{grad}\,\phi)$ for every C^∞-vector field X, and I_n designates the identity one. This definition of the operator D, originally due to N. Tanaka [8], can be proved to coincide with that given in (1), because we readily infer that the following formula holds:

(3) $$\mathrm{grad}\,\phi|_{S^n}=\mathrm{grad}\,\phi+\phi\cdot\xi,$$

where we denote by grad ϕ the usual gradient vector field of the function ϕ that is considered as the homogeneous function of degree one over $\boldsymbol{R}^{n+1}-\{0\}$ extended from $\phi\in C^\infty(S^n)$ as usual.

2. In this note, we shall be concerned with a *complex analogue* of the equation (2). Namely, denoting by $\boldsymbol{C}^{n+1}$ $(n\geq 2)$ the complex cartesian space with the complex co-ordinates $(\xi_0,\xi_1,\cdots,\xi_n)$, we consider the odd-dimensional unit sphere S^{2n+1} in $\boldsymbol{C}^{n+1}$ such that

$$S^{2n+1}=\{\xi=(\xi_0,\xi_1,\cdots,\xi_n)\in\boldsymbol{C}^{n+1};\ \textstyle\sum_{i=0}^n|\xi_i|^2=1\}.$$

Thus we may now consider the equation (2) with respect to the unknown function $\tilde{\phi}$ on S^{2n+1}, which we write as

(4) $$\tilde{D}(\tilde{\phi})=\tilde{\kappa},$$

for some (known) positive function $\tilde{\kappa}$.

Our situation in the present one is a somewhat special one: Namely let us take $\kappa\in C^\infty(P_C^n)$ and put $\tilde{\kappa}=\kappa\circ\pi\in C^\infty(S^{2n+1})$, where π denotes the natural projection of S^{2n+1} onto the complex projective space P_C^n, whose homogeneous co-ordinates being regarded as $(\xi_0,\xi_1,\cdots,\xi_n)$. Then $\pi: S^{2n+1}\to P_C^n$ is a principal C^∞-fibre bundle with the group $T^1=\{\theta\in\boldsymbol{C};\ |\theta|=1\}$, thus we have now to assume that

(5) $\tilde{\kappa}$ is invariant under the action of T^1, and positive everywhere.

From this condition we can infer easily the following relation:

(6) $$\int_{S^{2n+1}}\tilde{\kappa}\cdot\xi_i dS=0\qquad(0\leq i\leq n),$$

where dS denotes the volume element of S^{2n+1} as used in the preceding section. In fact, we see

$$\int_{S^{2n+1}}\tilde{\kappa}\cdot\xi_i dS=\int_{P_C^n}\kappa\left\{\int_{T_z}\xi_i(\xi t)dt\right\}dP,$$

where $T_z=\pi^{-1}(z)$ for $z\in P_C^n$ designates the fiber on $z\in P_C^n$ $(T_z\cong T^1)$, $t\in T_z$ and dP the volume element of P_C^n defined in the canonical manner. It is easily verified that the integral over T_z always vanishes for any z, thus (6) is derived from (5). Under the condition (5), we can show by using the so-called continuity method, as was utilized in [2, 4, 8], the following:

Theorem 1. *There is a unique C^∞-elliptic solution $\tilde{\phi}$ on S^{2n+1} which is invariant under the action of T^1 for a given positive function $\tilde{\kappa}$ satisfying the condition* (5).

From this theorem, it follows immediately our results for P^n_C; in fact, the differential operator $\tilde{D}$ turns out to induce in a natural manner the differential operator D on $C^\infty(P^n_C)$. Thus, if we consider the differential equation on P^n_C for any given positive function κ:

$$D(\phi)=\kappa, \tag{7}$$

then, we obtain as an immediate consequence of Theorem 1:

Theorem 2. *The equation* (7) *has the unique C^∞-elliptic solution ϕ for any given C^∞-positive function κ, whereby the solution ϕ is elliptic in the sense that the Gâteaux derivative dD_ϕ of D at ϕ is a linear elliptic differential operator* (see [7]).

3. We shall sketch here, and also in the next section, an outline of the proofs of Theorems 1, 2. For this sake, we need to consider the space $\tilde{F}$ of all C^∞-functions of $\tilde{M}=S^{2n+1}$ which are invariant under T^1, and further we have to introduce two Fréchet spaces with the C^∞-topology:

$$\tilde{K}=\{\tilde{\kappa}\in\tilde{F};\ \tilde{\kappa}>0\},$$
$$\tilde{K}^0=\{\tilde{\kappa}\in\tilde{K};\ (4)\text{ has a }T^1\text{-invariant solution }\tilde{\phi}\}.$$

Following the principle of the continuity method, we first prove the so-called openness (see [2, 4, 8]). Namely, putting $\tilde{E}=\{\tilde{\phi}\in\tilde{F};\ \tilde{\phi}\text{ is elliptic}\}$, we take $\tilde{\phi}\in\tilde{E}$, $\tilde{\kappa}\in\tilde{K}^0$ such that $\tilde{D}(\tilde{\phi})=\tilde{\kappa}$ as in (4). Then, we are now concerned with the partial differential equation with the unknown function $\tilde{\psi}$ on $\tilde{M}$: $\tilde{D}(\tilde{\phi}+\tilde{\psi})=\tilde{\kappa}'$, where $\tilde{\kappa}'\in\tilde{K}$ and $\tilde{\kappa}'-\tilde{\kappa}=\tilde{\kappa}''$ is assumed to be *so small* in the sense of the C^∞-topology; namely taking the difference we get

$$D(\tilde{\phi}+\tilde{\psi})-\tilde{D}(\tilde{\phi})=\tilde{\kappa}''.$$

The left-hand side of this equation can be decomposed into the two parts; the first one $\tilde{L}_{\tilde{\phi}}(\tilde{\psi})$ is defined as the linear part with respect to $\tilde{\psi}$, where $\tilde{L}_{\tilde{\phi}}$ is an elliptic linear differential operator acting on the space $\tilde{F}$ and coincides with the Gâteaux derivative $d\tilde{D}_{\tilde{\phi}}$ of $\tilde{D}$ at $\tilde{\phi}$, while the second one $\tilde{R}_{\tilde{\phi}}(\tilde{\psi})$ is the remaining term with the higher degree with respect to $\tilde{\psi}$. Thus we may rewrite the above as follows:

$$\tilde{L}_{\tilde{\phi}}(\tilde{\psi})=\tilde{\kappa}''-\tilde{R}_{\tilde{\phi}}(\tilde{\psi}). \tag{8}$$

Here, we are in a position to utilize the general theory of linear elliptic partial differential equations on $\tilde{M}$, and then the *iteration method*, which was carried out also in [4, 8]; namely we start from $\tilde{\psi}_0=0$ and then define inductively the C^∞-functions $\tilde{\psi}_k$ $(k=1, 2, \cdots)$ by a solution of the following equation:

$$\tilde{L}_{\tilde{\phi}}(\tilde{\psi}_k)=\tilde{\kappa}''-\tilde{R}_{\tilde{\phi}}(\tilde{\psi}_{k-1})(=\tilde{f}_k\in\tilde{F}) \qquad (k\geq 1),$$

in fact, $\tilde{\psi}_k$ is uniquely determined as

$$\tilde{\psi}_k(\xi)=\int_{\tilde{M}}\tilde{G}_{\tilde{\phi}}(\xi,\eta)\tilde{f}_k(\eta)\cdot d\tilde{M}_\eta.$$

In this formula, $\tilde{G}_{\tilde{\phi}}(\xi,\eta)$ denotes the Green function of the T^1-invariant elliptic operator $\tilde{L}_{\tilde{\phi}}$ with reference to the space $\tilde{F}$. Hence we infer obviously that the Green operator $\tilde{G}_{\tilde{\phi}}(\xi):\eta\to\tilde{G}_{\tilde{\phi}}(\xi,\eta)$, and also each $\tilde{\psi}_k$ belongs to $\tilde{F}$. The existence of a solution $\tilde{\psi}$ of (8), for a small κ'', is thus assured by the existence of the limit of the sequence $\{\psi_k\}$ as was done in [4,8]; in fact, if we take $\lim_{k\to\infty}\tilde{\psi}_k=\tilde{\psi}_\infty$, then $\tilde{\psi}_\infty$ evidently satisfies (8). This proves the *openness*.

4. To prove the so-called *closedness* in the continuity method [2, 4, 6]. We need, as was done for (2) in [4, 6], to establish the so-called *a priori estimate* of the solution $\tilde{\phi}$ of (4), or ϕ of (7). For this sake, we can utilize the result of A. V. Pogorolov, *via* the bundle diagram (9) in the next section; namely, for $\tilde{\phi}=\phi\circ\pi\in\tilde{E}$, we have $\|\tilde{\phi}\|_{m+\alpha}=\|\phi\|_{m+\alpha}$ in the sense of the usual $C^{m+\alpha}$-Banach norms. Thus, the closedness in our case is an immediate consequence of that in the Minkowski problem.

The uniqueness in Theorems 1, 2 is derived from the fact that, in the equation (2), the solutions are unique up to translations in $\boldsymbol{R}^{n+1}$ (see [1]), and also from that the solutions have to be invariant under T^1. As for this point, we should like to mention the results due to J. Moser and T. Sunada [7], which assert that, if the solution space of $dD_\phi(\psi)=0$ is of *0-dimension* for a given elliptic solution ϕ then so is for the solution space of (7).

5. We should now like to proceed to clarify the geometric meanings of the solution ϕ in (7), as was known for the solutions ϕ in (2); namely for the latter case, grad $\tilde{\phi}=(\partial\tilde{\phi}/\partial\xi_0,\cdots,\partial\tilde{\phi}/\partial\xi_n)$ yields a *convex* imbedding of S^n into $\boldsymbol{R}^{n+1}$ when one considers $\tilde{\phi}$ as the function extended over $\boldsymbol{R}^{n+1}-\{0\}$ as a homogeneous function of degree 1. For this purpose, we have here to recall the canonical (minimal) imbedding ι of $P^n_{\boldsymbol{C}}$ $(=M)$ into the space H^1_{n+1} $(=R^N)$ consisting of $(n+1)$-hermitian matrices with trace 1 $(N=n^2+2n)$. As is well-known (see [9]), the mapping ι is induced from the mapping $\tilde{\iota}$ of $\tilde{M}$ into H^1_{n+1} such that $\tilde{\iota}(\xi_0,\xi_1,\cdots,\xi_n)=(\bar{\xi}_i\xi_j)$ for $\xi=(\xi_0,\xi_1,\cdots,\xi_n)\in\tilde{M}$; we present explicitly these situations by the diagram given below:

$$(9)\qquad \begin{array}{ccc} \boldsymbol{C}^{n+1}\supset\tilde{M} & & \\ \downarrow\pi & \searrow\tilde{\iota} & \\ M & \xrightarrow{\ \iota\ } & \boldsymbol{R}^N=H^1_{n+1}. \end{array}$$

Now, we shall return to the solution ϕ of (7). For any C^∞-solution ϕ, we consider the lift $\tilde{\phi}=\phi\circ\pi$ and the hermitian matrix $(\tilde{\phi}_{\xi_i}\cdot\tilde{\phi}_{\xi_j})$, where $\tilde{\phi}_{\xi_i}=\partial\tilde{\phi}/\partial\xi_i$ and $\tilde{\phi}$ is extended over $\boldsymbol{C}^{n+1}-\{0\}$ as before. Then, this matrix-valued C^∞-function, defined on $\tilde{M}$ (restricted to $\tilde{M}$), is invariant under the action of T^1. Therefore it induces a C^∞-mapping ι_ϕ that is in fact an *imbedding* of P_C^n $(=M)$ into H_{n+1}^1 $(=\boldsymbol{R}^N)$:

$$\iota_\phi(\xi_0, \cdots, \xi_n) = \left[\frac{\tilde{\phi}_{\xi_i}\tilde{\phi}_{\xi_j}}{\|\mathrm{grad}\,\tilde{\phi}\|^2}\right]; \qquad \|\mathrm{grad}\,\tilde{\phi}\|^2 = \sum_{k=0}^{n} |\tilde{\phi}_{\xi_k}|^2.$$

Namely ι_ϕ gives rise to a family of imbeddings of P_C^n into H_{n+1}^1. On the other hand, we know that the canonical imbedding ι is known to be *elliptic* in the sense of Tanaka [9], thus we suppose that there will be certain intimate relations between the positive function κ in (7) and some kind of curvature of $\iota_\phi(M)$ in $\boldsymbol{R}^N$. We hope that we shall be able to discuss in the near future with these geometric problems, not only over P_C^n but also over more general compact spaces.

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